Numerical Solution of Fractional Order Partial Differential Equation with Sturm-Liouville Problem Using Homotopy Analysis and Homotopy Perturbation Methods

*B. M. Yisa[1] and N. A. Adelabu[2]
[1,2]Department of Mathematics, Faculty of Physical Sciences, University of Ilorin

Email: yisa.bm@unilorin.edu.ng

Abstract: This research work is concerned with the application of both homotopy analysis and homotopy perturbation methods to linear and nonlinear, homogeneous, and nonhomogeneous fractional order partial differential equations and fractional order Sturm-Liouville equation. The fractional order derivatives are interpreted in Caputo sense. The applications of the two semi analytical methods are extended to one dimensional fractional order wave equation. Although homotopy perturbation method involves asymptotic expansion of terms with small parameter, but it pays off in the accuracy of the results obtained through which are similar to results using homotopy analysis method. All the problems selected from the existing literature made provision for results with integral order values, but in the present work we equally present results for fractional order. Our results are presented in 3D graphs and compared well with the existing results in the literature.

Keywords: *Embedding Parameter, Homotopy Maclaurin Expansion, Asymptotic Expansion, Zeroth-order Deformation, nth-order Deformation, Perturbation Parameter*.

1. INTRODUCTION

In 1992, Liao in his PhD thesis used the basic ideas of homotopy in topology to propose a general analytical method for nonlinear problems. He named his new method Homotopy analysis method (HAM) which he used to obtain the series solution of nonlinear differential equations [2,3,36]. In the last few decades, the HAM has been applied to solve many types of nonlinear problems [10,13,33]. Homotopy analysis method makes use of an auxiliary parameter which provides a convenient way to control and adjust the convergence rate of approximation [3].

The Homotopy Perturbation Method (HPM) which harmonizes the perturbation method and homotopy method, was proposed by He [24]. It converts strictly nonlinear problems to the type that can easily be solved with reduced rigour [24,25,26]. HPM was also used in solving not only nonlinear initial value problems (IVPs), but also boundary value problems [27]. It suffices to state that without adequate knowledge of use perturbation method (PM), implementation of HPM would be an illusion. [18] applied PM to solve non-periodic dynamical system problem. Although the method was originally introduced much earlier and used by many scientist [2].

In recent years, differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid, mechanics, visco elasticity, biology physics and engineering [21]. Consequently, considerable attention has been given to the solutions of fractional differential and integral equations of physical interest [3]. Many important phenomena in electromagnetism, acoustics, viscoelasticity, electrochemistry, and material science are well described by differential equations of fractional order [17]. Fractional theory has many applications [5]. Moreover, fractional modelling has been applied in micro-grids [4] and decentralized wireless network [3].

Fractional differential equations involve real or complex order derivatives [9], various researchers contributed on fractional derivatives during the 18th and 19th centuries for example Abel, Caputo, Euler, Fourier, Laplace, Liouville and Ross [4].

Most of the problems in science and engineering, physics, chemistry, and other sciences are nonlinear and can be described very successfully by modelling using fractional calculus [33]. That is, the theory of derivatives and integrals fractional order [20]. For instance, the nonlinear oscillation of earthquakes can be model with fractional derivatives can eliminate the deficiency arising in the assumption of continuum traffic flow [22]. Most fractional differential equations do not have exact solutions, so the approximate and numerical techniques must be used [15]. Various analytical method such as Variational Iteration Method, Adomian Decomposition Method, Homotopy Analysis Method, and Homotopy Perturbation Method have been used to solve several forms of differential equations [19,24,26,29,30,35]

In this work, two semi-analytical methods used are HAM and HPM. These two methods are selected for the solutions of nonlinear, homogeneous, and inhomogeneous fractional order partial differential equations, because of their accuracy, simplicity in implementation and low computer memory utilization. With the simplified algorithms tailored after the well-known HAM and HPM for the class of problems we solved, that are present in sections 4.2 and 5.2, we were able to obtain solutions for both fractional and integer orders. We equally present 3D graphs for the two cases.

2. PRELIMINARIES

2.1 Fractional Calculus

The field of fractional calculus is almost as old as calculus itself, but over the last few decades the usefulness of this mathematical theory and its applications as well as its merits in the field mathematics has become more evident. Several definitions were evolved, some modifying some existing ones. Few important ones used in this work are given in the sequel.

2.1.1 The Riemann-Liouville Fractional Derivative

The Riemann-Liouville integral of order $\eta \geq 0$ for continuous function f on $[a, b]$ is defined by

$$J^{\eta} f(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)\, dt, \quad \eta > 0, \qquad (1)$$

$a < x < b$, where Gamma function of η is

$$\Gamma(\eta) = \int_0^{\infty} e^{-x} x^{\eta-1}\, dx. \qquad (2)$$

And the result in (1) can equivalently be written as

$$J^{\eta} t^{\xi} = \frac{\Gamma(\xi+1)}{\Gamma(\xi-\eta+1)} t^{\xi+\eta}. \qquad (3)$$

2.1.2 *The Caputo Fractional Derivative*

The Caputo fractional derivative operator D^{η} of order η is defined as follows

$$D^{\eta} f(t) = \frac{1}{\Gamma(n-\eta)} \int_0^x f^{(n)}(t)(x-t)^{n-\eta-1}\, dt, \qquad (4)$$

where $r - 1 < \eta < r,\ r \in N, \eta > 0$ and $x > 0$.

This can as well be written as

$$D^{\eta} t^{\xi} = \frac{\Gamma(\xi+1)}{\Gamma(\xi-\eta+1)} t^{\xi-\eta}. \qquad (5)$$

2.2 *Properties of Homotopy Derivative*

Given that ϕ and φ are two functions that are analytic with respect to the homotopy parameter $p \epsilon [0, 1]$ in a manner that $\phi = \sum_{j=0}^{\infty} u_j p^j$ and $\varphi = \sum_{j=0}^{\infty} u_j p^j$ represent their respective Maclaurin series. The rth homotopy derivative of ϕ is given as

$$D_r(\phi) = \frac{1}{r!} \frac{d^r \phi}{dp^r} \Big|_{p=0}, \qquad r \geq 0, r \epsilon Z \qquad (6)$$

Furthermore, if $D_r(\phi) = u_r$, it follows that

$$D_r(p^j \phi) = D_{r-j}(\phi) = u_{r-j}, \qquad 0 < j < r, \quad (7)$$

and

$$D_r(p^j \phi\varphi) = \sum_{j=0}^{r} D_j(\phi)\, D_{r-j}(\varphi) = \sum_{j=0}^{r} u_{r-j}\, u_j. \ (8)$$

See [29,30,31].

3 Statement of Problem

The three classes of fractional order partial differential equations considered are:

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \gamma \frac{\partial^2 u(x,t)}{\partial x^2} + \xi \frac{\partial u(x,t)}{\partial t} + \lambda u(x,t) \ (9)$$

where γ, $\xi$and λ are non-negative constants, and $1 < \alpha < 2$. The associated boundary conditions are $u(0,t) = pf(t)$, $u_x(0,t) = qf(t)$ and $u(x,0) = rf(x)$, where $p, q, r \epsilon R$;

$$u_t^{\mu} u(x,t) = k \frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}} + t, \qquad (10)$$

with associated initial and boundary conditions

$$u(0,t) = u(\pi,t) = 0, 1 \leq \mu \leq 2, \gamma = 2$$
$$u(x,0) = x,\ 0 \leq x \leq \pi$$
$$u_t(x,0) = 0,\ 0 \leq x \leq \pi,$$

and fractional order Sturm-Liouville equation $D^{\mu} y(t) + p(t) y(t) = \lambda \phi y(t),\ 1 \leq \mu \leq 2, \quad (11)$

subject to

$cy(0) + dy'(0) = 0$ and $ry(1) + sy'(1) = 0$,

where $c, d, r, s \in R$.

4 *A Review of Homotopy Analysis Method (HAM)*

4.1 The Zeroth and rth Order Deformation Equation

Consider the nonlinear differential equation

$$N[u(x)] = 0, \qquad (12)$$

where N is a nonlinear differential operator, x is the independent variable, and $u(x)$ the unknown function. There are essentially two types of deformation in HAM, and they are as given

$$(1-p) L[\phi(x;p) - u_0(x)] = phN[\phi(x;p)], \quad (13)$$

where $p \in [0, 1]$ is an embedding parameter, $h \neq 0$, L is an auxiliary linear operator, u_0 is an initial guess of $u(x)$ which generally satisfies the initial conditions, and $\phi(x, p)$ is an unknown function.

While the nth order deformation takes the form

$$L[u_r(x) - \chi_r u_{r-1}(x)] = hD_{r-1}\left[N\left(\phi(x;p)\right)\right], (14)$$

where

$$\chi_r = \begin{cases} 0, & r \leq 1 \\ 1, & r > 1. \end{cases}$$

(18) is obtained after r times differentiation of zeroth order deformation equation.

The $D_{r-1}\left[N\left(\phi(x;p)\right)\right]$ is given by

$$D_{r-1}\left[N\left(\phi(x;p)\right)\right] = \frac{1}{(r-1)!} \frac{\partial^{r-1} N[\phi(x;p)]}{\partial p^{r-1}} \ (15)$$

In this research, $h = -1$ in most cases and in few cases $h = 1$ (depending on the value adopted in the literature for the specific problem).

4.2 *Methodology: Implementation of HAM*

The problem in (9) is discussed here with the details of implementation of HAM.

The homotopy analysis method requires that

$$L(x) = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} \qquad (16)$$

and

$$N(x) = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \gamma \frac{\partial^2 u(x,t)}{\partial t^2} - \xi \frac{\partial u(x,t)}{\partial t} - \lambda u(x,t). \qquad (17)$$

Using the mth order deformation equation

$$L(u_m - \chi_m u_{m-1}) = hD_{m-1}N[u], \qquad (18)$$

and the homotopy derivative D_m, we have

$$D_m[N[u]] = D_m\left[\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \gamma \frac{\partial^2 u(x,t)}{\partial t^2} - \xi \frac{\partial u(x,t)}{\partial t} - \lambda u(x,t)\right]$$

$$= \left[D_m \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \gamma D_m \frac{\partial^2 u(x,t)}{\partial t^2} - \xi D_m \frac{\partial u(x,t)}{\partial t} - \lambda D_m u(x,t)\right]$$

$$= \frac{\partial^\alpha u_m(x,t)}{\partial x^\alpha} - \gamma \frac{\partial^2 u_m(x,t)}{\partial t^2} - \xi \frac{\partial u_m(x,t)}{\partial t} - \lambda u_m(x,t), \quad m = 0,1,2,.. \quad (19)$$

Thus, the successive derivatives take the form

$$D_0[N[u]] = \frac{\partial^\alpha u_0}{\partial x^\alpha} - \gamma \frac{\partial^2 u_0}{\partial t^2} - \xi \frac{\partial u_0}{\partial t} - \lambda u_0$$

$$D_1[N[u]] = \frac{\partial^\alpha u_1}{\partial x^\alpha} - \gamma \frac{\partial^2 u_1}{\partial t^2} - \xi \frac{\partial u_1}{\partial t} - \lambda u_1$$

$$D_2[N[u]] = \frac{\partial^\alpha u_2}{\partial x^\alpha} - \gamma \frac{\partial^2 u_2}{\partial t^2} - \xi \frac{\partial u_2}{\partial t} - \lambda u_2 \qquad (20)$$

Using (20) in (18), we have for m=1, where $\chi_1 = 0$,

$$L(u_1 - \chi_1 u_0) = hD_0 N[u], \qquad (21)$$

implies

$$L(u_1) = h\left[\frac{\partial^\alpha u_0}{\partial x^\alpha} - \gamma \frac{\partial^2 u_0}{\partial t^2} - \xi \frac{\partial u_0}{\partial t} - \lambda u_0\right] \qquad (22)$$

$$\frac{\partial^\alpha u_0}{\partial x^\alpha} = h\left[\frac{\partial^\alpha u_0}{\partial x^\alpha} - \gamma \frac{\partial^2 u_0}{\partial t^2} - \xi \frac{\partial u_0}{\partial t} - \lambda u_0\right] \qquad (23)$$

At this juncture, $u(x,t)$ shall be expanded in Taylor series thus,

$$u(x,t) = u(a,b) + u_x(a,b)(x-a) + u_t(a,b)(t-b) + u_{xx}(a,b)\frac{(x-a)^2}{2!} + u_{tt}(a,b)\frac{(t-a)^2}{2!} + \cdots \quad (24)$$

Boundary conditions shall be substituted in (24) and u_0 shall be obtained as

$$u(x,t) = pf(t) + qf(t)(x-a) + pf'(t)(t-t) + 0 + \cdots$$

$$u(x,t) = f(t)[p + qx]$$

$$u_0 = f(t)[p + qx].$$

Since (23) can equivalently be written as

$$D^\alpha u_1 = h\left[D^\alpha u_0 - \gamma \frac{\partial^2 u_0}{\partial t^2} - \xi \frac{\partial u_0}{\partial t} - \lambda u_0\right], \qquad (25)$$

to get u_1 from (25), $D^{-\alpha}$ shall be applied to both sides of (25) thus

$$u_1 = hD^{-\alpha}\left[D^\alpha u_0 - \gamma \frac{\partial^2 u_0}{\partial t^2} - \xi \frac{\partial u_0}{\partial t} - \lambda u_0\right] \qquad (26)$$

$$u_1 = h\left[u_0 - \gamma D^{-\alpha} \frac{\partial^2 u_0}{\partial t^2} - \xi D^{-\alpha} \frac{\partial u_0}{\partial t} - \lambda D^{-\alpha} u_0\right] \qquad (27)$$

Substituting for u_0 in (27) gives

$$u_1(x,t) = h\left[f(t)[p+qx] - \gamma f''(t)\left[p\frac{x^\alpha}{\Gamma(\alpha+1)} + q\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right] - \xi f'(t)\left[p\frac{x^\alpha}{\Gamma(\alpha+1)} + q\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right] - \lambda f(t)\left[p\frac{x^\alpha}{\Gamma(\alpha+1)} + q\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right] \qquad (28)$$

The result for the subsequent values of m shall be obtained in like manners.

5 *Review of HPM Algorithm*

The homotopy equation corresponding to problem (9) is given as

$$(1-p)L[u(x) - u_0(x)] = pN[u(x)] \qquad (29)$$

where p and u_0 have their usual meaning.

The embedding parameter p here serves as the perturbation parameter in the asymptotic expansion of $u(x,p)$ which is defined as

$$u(x,p) = \sum_{k=0}^{r} p^k u_k + O(p^{r+1}). \qquad (30)$$

The solution to the problem is

$$u(t) = u_0(t) + pu_1(t) + p^2 u_2(t) + \cdots \qquad (31)$$

where u_j $(j = 0,1,2,3,\ldots)$ **are functions to be determined.**

5.1 Methodology: Implementation of HPM

Implementation of homotopy perturbation method (HPM) is illustrated below.

Let $p > 0$ be the small perturbation parameter so that the asymptotic expansion of $\frac{\partial u(x,p)}{\partial x}$ be

$$\frac{\partial u(x,p)}{\partial x} = \sum_{k=0}^{r} p^k \frac{\partial u_k}{\partial x} + O(p^{r+1}) \qquad (32)$$

$$\frac{\partial u(x,p)}{\partial x} = \frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + p^2\frac{\partial u_2}{\partial x} + O(p^{r+1}) \quad (33)$$

The homotopy perturbation equivalent of (9) is

$$\frac{\partial^\alpha u}{\partial x^\alpha} = p\left[\gamma\frac{\partial^2 u}{\partial t^2} + \xi\frac{\partial u}{\partial t} + \lambda u - D^\alpha\right] \quad (34)$$

Expanding both sides of (34) asymptotically gives

$$\frac{\partial^\alpha u_0}{\partial x^\alpha} + p\frac{\partial^\alpha u_1}{\partial x^\alpha} + p^2\frac{\partial^\alpha u_2}{\partial x^\alpha} + \ldots = p[\gamma\frac{\partial^2 u_0}{\partial t^2} + p\gamma\frac{\partial^2 u_0}{\partial t^2} + p^2\gamma\frac{\partial^2 u_0}{\partial t^2} + \cdots + \xi\frac{\partial u_0}{\partial x} + p\xi\frac{\partial u_1}{\partial x} + p^2\xi\frac{\partial u_2}{\partial x} + \cdots + \lambda u_0 + p\lambda u_1 + p^2\lambda u_2 + \cdots - D_x^\alpha u_0] \quad (35)$$

Comparing both sides of (35) for the coefficients of various powers of the perturbation parameter p, we have

$$p^0: \frac{\partial^\alpha u_0}{\partial x^\alpha} = 0$$

$$p^1: \frac{\partial^\alpha u_1}{\partial x^\alpha} = \gamma\frac{\partial^2 u_0}{\partial t^2} + \xi\frac{\partial u_0}{\partial t} + \lambda u_0 - D_x^\alpha u_0$$

$$p^2: \frac{\partial^\alpha u_2}{\partial x^\alpha} = \gamma\frac{\partial^2 u_1}{\partial t^2} + \xi\frac{\partial u_1}{\partial t} + \lambda u_1$$

$$p^3: \frac{\partial^\alpha u_3}{\partial x^\alpha} = \gamma\frac{\partial^2 u_2}{\partial t^2} + \xi\frac{\partial u_2}{\partial t} + \lambda u_2 \quad (36)$$

The system of equations in (36) are then solved using the Taylor series in (24) and the boundary conditions.

6 *Numerical Experiment*

In this section, selected problems from literature are solved by both Homotopy Analysis Method and Homotopy Perturbation Method.

Problem 1

Consider the fractional order homogeneous partial differential equation

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial u(x,t)}{\partial t} - u(x,t) = 0,$$

with the initial and boundary conditions,

$u(0,t) = e^{-t},\ u_x(0,t) = e^{-t}, u(x,0) = e^x.$

Solution Based on Homotopy Analysis Method (HAM)

Implementing the algorithm presented in Section 4, the following solutions are obtained:

$u_0 = e^{-t}(x+1).$

$$u_1 = h\left[e^{-t} + xe^{-t} - \left[\frac{e^{-t}}{\Gamma(\alpha+1)}x^\alpha + \frac{e^{-t}}{\Gamma(\alpha+2)}x^{\alpha+1}\right]\right]$$

$$u_2 = (h+1)u_1 - h^2 e^{-t}\left[\frac{e^{-t}}{\Gamma(\alpha+1)}x^\alpha + \frac{e^{-t}}{\Gamma(\alpha+2)}x^{\alpha+1} + \frac{e^{-t}}{\Gamma(2\alpha+1)}x^{2\alpha}\right]$$

When $h = -1$ and $\alpha = \frac{1}{2}$,

$u_0 = e^{-t}(x+1)$

$u_1 = e^{-t}[\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} - 1 - x]$

$u_2 = e^{-t}[x + \frac{x^2}{2!} - \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} - \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}]$

and the series approximate solution is

$u(x) \approx [x + \frac{x^2}{2!} + \cdots]e^{-t}.$

To obtain the solution in the referenced literature, we shall set $\alpha = 2$ and $h = -1$. Thus, we have

$u_0 = e^{-t}(x+1).$

$$u_1 = -e^{-t}(x + 1 - \frac{x^2}{2!} - \frac{x^3}{3!})$$

$$u_2 = -e^{-t}(\frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{x^5}{5!})$$

The final series approximate solution is

$$u(x) \approx e^{-t}\left(\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right).$$

Solution Based on Homotopy Perturbation Method

Implementing the algorithm presented in section 5, the following solutions are obtained:

$$u_0 = e^{-t}(x+1).$$

$$u_1 = e^{-t}[\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - (1+x)]$$

$$u_2 = e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]$$

When $\alpha = \frac{1}{2}$, we have

$$u_0 = e^{-t}(x+1).$$

$$u_1 = e^{-t}[\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}} - 1 - x]$$

$$u_2 = e^{-t}[x + \frac{x^2}{2!} - \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} - \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}]$$

Thus, the series approximate solution is

$u(x) \approx [x + \frac{x^2}{2!} + \cdots]e^{-t}.$

When $\alpha = 2$, we have

$$u_0 = e^{-t}(x+1)$$

$$u_1 = e^{-t}(\frac{x^2}{2!} + \frac{x^3}{3!} - 1 - x)$$

$$u_2 = e^{-t}(\frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^3}{3!} - \frac{x^2}{2!})$$

in which case the series approximate solution is

$$u(x) \approx e^{-t}\left(\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right).$$

Problem 2

Consider the fractional order nonlinear homogeneous partial differential equation

$u_t^\alpha - u_{xxt} + u_x - uu_{xx} + uu_x - 3u_x u_{xx} = 0$,

with the initial condition

$$u(x,0) = \frac{8}{3}e^{\frac{x}{2}}, \qquad t \geq 0, \qquad 0 \leq \alpha \leq 1.$$

Solution Based on Homotopy Analysis Method

$$u_0 = \frac{8}{3}e^{\frac{x}{2}}$$

$$u_1 = \frac{4h}{3}e^{\frac{x}{2}}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2 = (1+h)u_1 - \frac{h^2}{3}e^{\frac{x}{2}}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{2h^2}{3}e^{\frac{x}{2}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

When $h = -1$ and $\alpha = \frac{1}{2}$

$$u_0 = \frac{8}{3}e^{\frac{x}{2}}$$

$$u_1 = -\frac{8}{3}e^{\frac{x}{2}}\frac{t^{\frac{1}{2}}}{\sqrt{\pi}}$$

$$u_2 = -\frac{1}{3}e^{\frac{x}{2}} + \frac{2t}{3}e^{\frac{x}{2}}$$

and the series approximate solution is

$$u \approx e^{\frac{x}{2}}[\frac{7}{3} - \frac{8}{3}\frac{t^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{2t}{3} + \cdots].$$

To obtain the solution in the referenced literature, we shall set $\alpha = 1$ and $h = -1$. Thus, we have

$$u_0 = \frac{8}{3}e^{\frac{x}{2}}$$

$$u_1 = -\frac{4t}{3}e^{\frac{x}{2}}$$

$$u_2 = -\frac{t}{3}e^{\frac{x}{2}} + \frac{t^2}{3}e^{\frac{x}{2}}$$

Hence, the series approximate solution is

$$u \approx \frac{8}{3}e^{\frac{x}{2}} - \frac{5t}{3}e^{\frac{x}{2}} + \frac{t^2}{3}e^{\frac{x}{2}} + \cdots.$$

Solution Based on Homotopy Perturbation Method

$$u_0 = \frac{8}{3}e^{\frac{x}{2}}$$

$$u_1 = -\frac{4}{3}e^{\frac{x}{2}}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2 = -\frac{1}{3}e^{\frac{x}{2}}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{2}{3}e^{\frac{x}{2}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

For $\alpha = \frac{1}{2}$, we have

$$u_0 = \frac{8}{3}e^{\frac{x}{2}}$$

$$u_1 = -\frac{4}{3}e^{\frac{x}{2}}\frac{t^{\frac{1}{2}}}{\sqrt{\pi}}$$

$$u_2 = -\frac{1}{3}e^{\frac{x}{2}} + \frac{2t}{3}e^{\frac{x}{2}}$$

Thus, the series approximate solution is

$$u \approx e^{\frac{x}{2}}[\frac{7}{3} - \frac{8}{3}\frac{t^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{2t}{3} + \cdots].$$

For $\alpha = 1$, we have

$$u_0 = \frac{8}{3}e^{\frac{x}{2}}$$

$$u_1 = -\frac{4t}{3}e^{\frac{x}{2}}$$

$$u_2 = -\frac{t}{3}e^{\frac{x}{2}} + \frac{t^2}{3}e^{\frac{x}{2}}$$

which gives the series approximate solution

$$u \approx \frac{8}{3}e^{\frac{x}{2}} - \frac{5t}{3}e^{\frac{x}{2}} + \frac{t^2}{3}e^{\frac{x}{2}} + \cdots.$$

Problem 3

Consider the fractional order nonlinear homogeneous partial differential equation

$D_t^\alpha w - D_t^\alpha w_{xx} + w_x - ww_{xx} - 3(w_x w_{xx} - ww_{xxx}) = 0$,

with the initial condition

$$w(x,0) = -x,$$

Solution Based on Homotopy Analysis Method

$$w_0 = -x$$

$$w_1 = h(1-x)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$w_2 = (h+1)w_1 + 2h^2(1-x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

When $h = -1$ and $\alpha = \frac{3}{2}$

$$w_0 = -x$$

$$w_1 = \frac{4(1-x)t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$w_2 = \frac{(1-x)t^3}{3}$$

Thus, the series approximate solution is

$$w \approx -x + \frac{4(1-x)t^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{(1-x)t^3}{3} + \cdots$$

To obtain the solution in the referenced literature, we shall set $\alpha = 1$ and $h = -1$. Thus, we have

$$w_0 = -x$$

$$w_1 = t(1-x)$$

$$w_2 = t^2(1-x)$$

The series approximate solution is

$$w = -x + t(1-x) + t^2(1-x) + \cdots$$

Solution Based on Homotopy Perturbation Method

$$w_0 = -x$$

$$w_1 = (1-x)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$w_2 = 2(1-x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

When $\alpha = 1$, we have

$$w_0 = -x$$
$$w_1 = (1-x)t$$
$$w_2 = (1-x)t^2$$

The series solution is

$$w = -x + t(1-x) + t^2(1-x) + \cdots$$

At $\frac{3}{2}$,

$$w_0 = -x$$
$$w_1 = \frac{4(1-x)t^{\frac{3}{2}}}{3\sqrt{\pi}}$$
$$w_2 = \frac{(1-x)t^3}{3}$$

Which gives the series approximate solution as

$$w \approx -x + \frac{4(1-x)t^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{(1-x)t^3}{3} + \cdots$$

Problem 4

Consider the fractional order Sturm-Liouville problem

$$D^{\frac{3}{2}}y(t) + \lambda y(t) = 0,$$

with boundary condition

$$y'(0) = 0, \quad y(1) = 0.$$

Solution Based on Homotopy Analysis Method

$$y_0 = 1$$
$$y_1(t) = \lambda \frac{ht^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} = \frac{2\lambda h t^{\frac{3}{2}}}{\sqrt{\pi}}$$

$$y_2(t) = (h+1)y_1 + \frac{\lambda^2 t^3}{\Gamma(4)} = (h+1)y_1 + \frac{\lambda^2 t^3}{3!}$$
$$y_3(t) = (h+1)^2 y_1 + 2(h+1)\lambda^2 h^2 \frac{t^3}{\Gamma(4)} + \lambda^3 h^3 \frac{t^{\frac{9}{2}}}{\Gamma\left(\frac{11}{2}\right)}$$
$$y_3(t) = (h+1)^2 y_1 + 2(h+1)\lambda^2 h^2 \frac{t^3}{3!} + 32\lambda^3 h^3 \frac{t^{\frac{9}{2}}}{945\sqrt{\pi}}$$

When $h = -1$ and $\lambda = 1$, we have

$$y_0 = 1$$
$$y_1(t) = -\frac{2t^{\frac{3}{2}}}{\sqrt{\pi}}$$

$$y_2(t) = \frac{t^3}{3!}$$
$$y_3(t) = -\frac{32t^{\frac{9}{2}}}{945\sqrt{\pi}}$$

Thus, the series approximate solution is

$$y(t) \approx 1 - \frac{2t^{\frac{3}{2}}}{\sqrt{\pi}} + \frac{t^3}{3!} - \frac{32t^{\frac{9}{2}}}{945\sqrt{\pi}} + \cdots$$

Solution Based on Homotopy Perturbation Method

$$y_0 = 1$$
$$y_1(t) = -\lambda \frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} = -\frac{2\lambda t^{\frac{3}{2}}}{\sqrt{\pi}}$$
$$y_2(t) = \lambda^2 \frac{t^3}{\Gamma(4)} = \lambda^2 \frac{t^3}{3!}$$
$$y_3(t) = -\lambda^3 \frac{t^{\frac{9}{2}}}{\Gamma\left(\frac{11}{2}\right)} = -32\lambda^3 \frac{t^{\frac{9}{2}}}{945\sqrt{\pi}}$$

At $\lambda = 1$, we have

$$y_0 = 1$$
$$y_1(t) = -\frac{2t^{\frac{3}{2}}}{\sqrt{\pi}}$$
$$y_2(t) = \frac{t^3}{3!}$$
$$y_3(t) = \frac{-32t^{\frac{9}{2}}}{945\sqrt{\pi}}$$

The series approximate solution is

$$y(t) \approx 1 - \frac{2t^{\frac{3}{2}}}{\sqrt{\pi}} + \frac{t^3}{3!} - \frac{32t^{\frac{9}{2}}}{945\sqrt{\pi}} + \cdots$$

Problem 5

Consider the fractional order inhomogeneous diffusion-wave equation

$$D_t^{\alpha} u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + t$$

with initial and boundary conditions

$$u(0,t) = u(\pi,t) = 0, \qquad 1 \le \alpha \le 2,$$
$$u(x,0) = x, \ 0 \le x \le \pi,$$
$$u_t(x,0) = 0, \ 0 \le x \le \pi.$$

Solution Based on Homotopy Analysis Method

$$T_{n0} = B_n, \quad n = 1,2,3,\ldots$$
$$T_{n1} = hkn^2 B_n \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$T_{n2} = (h+1)T_{n1} + h^2(kn^2)^2 B_n \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$T_{n3} = (h+1)T_{n2} + h(h+1)kn^2 T_{n1} + h^3(kn^2)^3 B_n \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

At $h = -1$ and $\alpha = 1$, we have

$$T_{n0} = B_n$$
$$T_{n1} = -kn^2 B_n t$$
$$T_{n2} = (kn^2)^2 B_n \frac{t^2}{2!}$$

$$T_{n3} = -(kn^2)^3 B_n \frac{t^3}{3!}$$

Solution for inhomogeneous part:

$$H_{n1} = h[kn^2 h_n \frac{t^\alpha}{\Gamma(\alpha+1)} - M_n \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}]$$

$$H_{n2} = (h+1)H_{n1} + h^2(kn^2)^2 h_n \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - h^2 kn^2 M_n \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

When $h = -1$ and $\alpha = 1$, we have

$$H_{n1} = -kn^2 h_n t + M_n \frac{t^2}{2!}$$

$$H_{n2} = (kn^2)^2 h_n \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - kn^2 M_n \frac{t^3}{3!},$$

where

$$M_n = 2[\frac{1-(-1)^n}{n\pi}]$$

and

$$h_n = \frac{1}{\pi}\left(\frac{sinn\pi}{n^2} - \frac{cosnx}{n}\right).$$

The general solution to the problem is obtained using

$$u(x,t) = X(x)T(t)$$

where

$$T_n(t) = B_n - kn^2 B_n t + (kn^2)^2 B_n \frac{t^2}{2!} - (kn^2)^3 B_n \frac{t^3}{3!} \dots - kn^2 h_n t + M_n \frac{t^2}{2!} + (kn^2)^2 h_n \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - kn^2 M_n \frac{t^3}{3!}$$

and

$$X(x) = A_n sinnx, n = 1, 2, 3, \dots$$

Hence

$$u(x,t) = [A_n sinnx][B_n - kn^2 B_n t + (kn^2)^2 B_n \frac{t^2}{2!} - (kn^2)^3 B_n \frac{t^3}{3!} \dots - kn^2 h_n t + M_n \frac{t^2}{2!} + (kn^2)^2 h_n \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - kn^2 M_n \frac{t^3}{3!}]$$

At $n = 1, k = 1$, we have

$$h_1 = \frac{1}{\pi}[sin\pi - cosx]$$

and

$$M_1 = \frac{4}{\pi}.$$

Thus, the general solution becomes

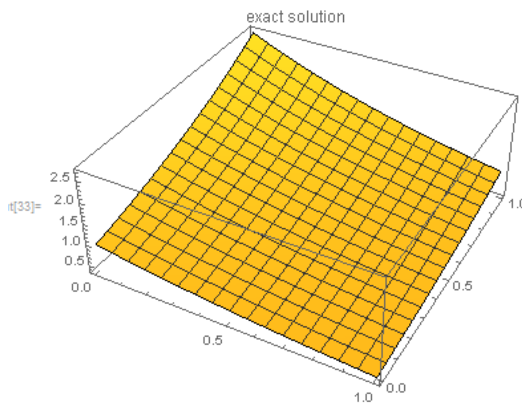
$$u(x,t) = [A_n sinx][B_n\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) + \frac{1}{\pi}(sin\pi - cosx)(-t + t^2 + \cdots) + \frac{4}{\pi}(\frac{t^2}{2!} - \frac{t^3}{3!} + \cdots)]$$

which is equivalent to

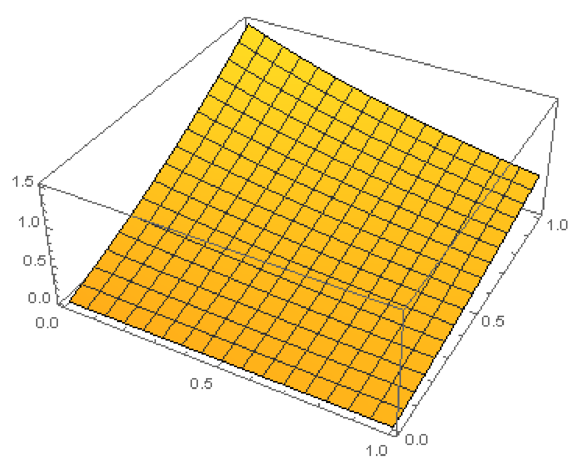
$$u(x,t) = [A_n sinx][B_n e^{-t} + \frac{1}{\pi}(sin\pi - cosx)(e^{-t} - 1) + \frac{4}{\pi}(e^{-t} - 1 + t)]$$

Similar result is obtained for the HPM.

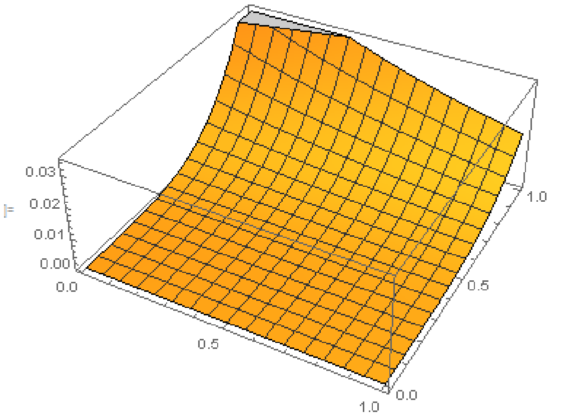
7 3D Representation for the Exact, HAM and HPM Result



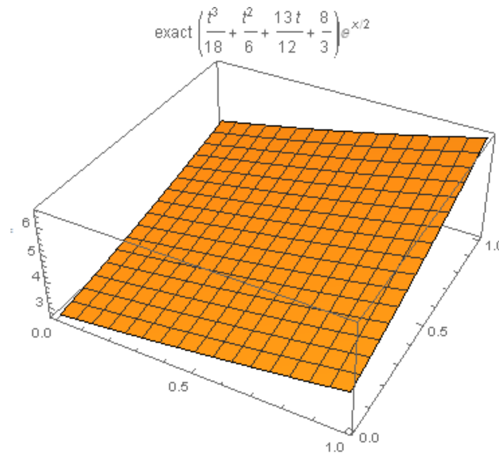
Problem 1: Exact Solution for $\alpha = 2$



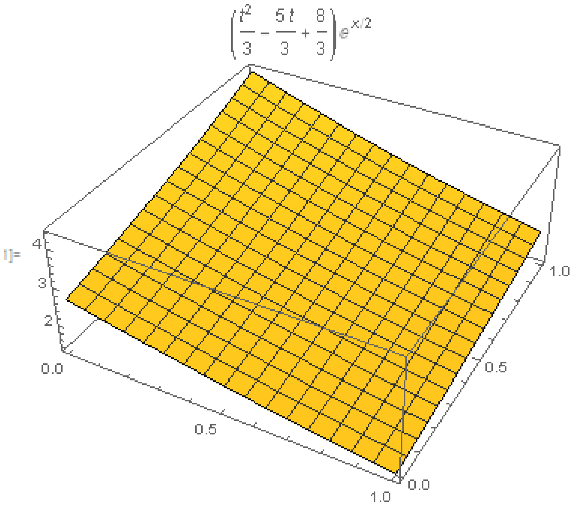
Problem 1: Approximate Solution by HAM and HPM for $\alpha = \frac{1}{2}$



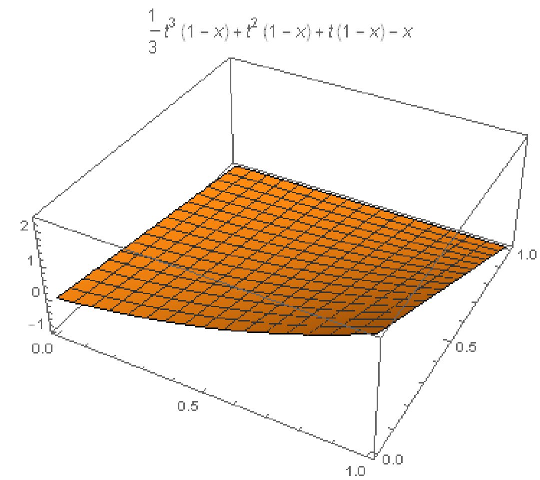
Problem 1: Approximate Solution by HAM and HPM for $\alpha = 2$



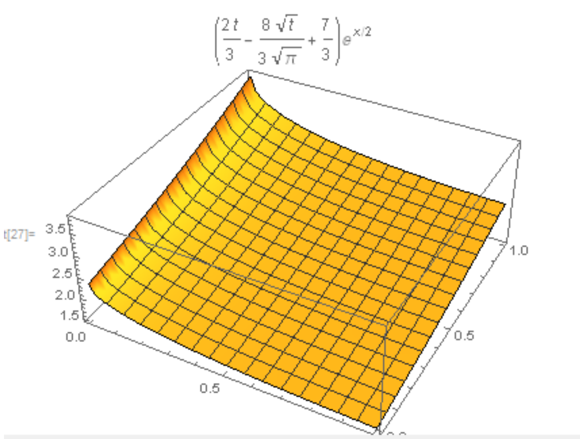
Problem 2: Exact Solution for $\alpha = 2$



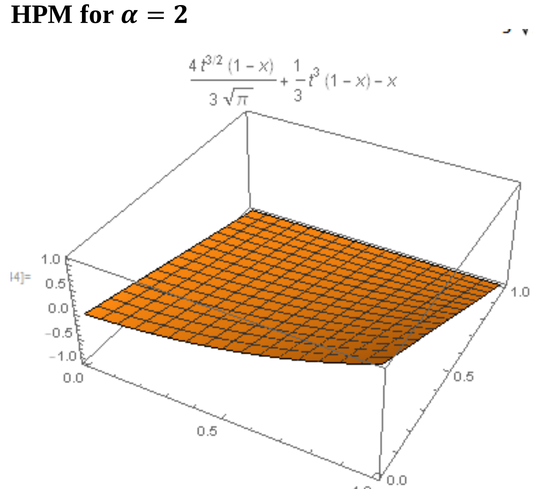
Problem 2: Approximate Solution by HAM and HPM for $\alpha = 2$



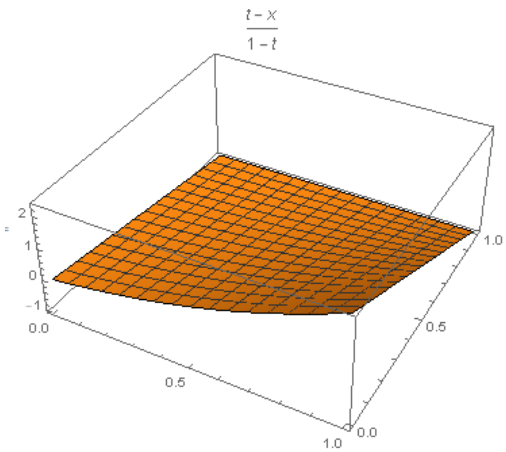
Problem 3: Approximate Solution by HAM and HPM for $\alpha = 2$



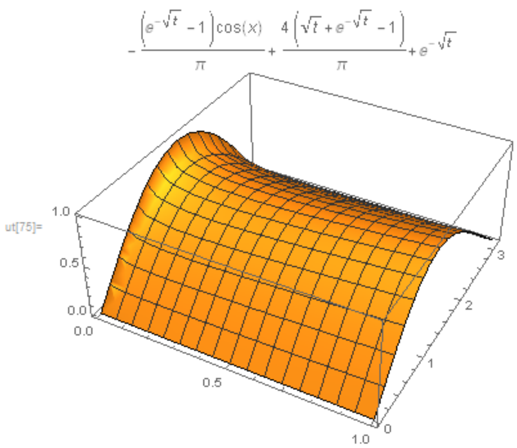
Problem 2: Approximate Solution by HAM and HPM for $\alpha = \frac{1}{2}$



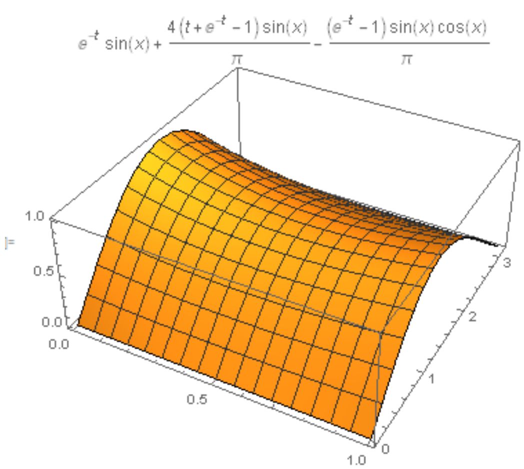
Problem 3: Approximate Solution by HAM and HPM for $\alpha = \frac{1}{2}$



Problem 3: Exact Solution for $\alpha = 2$



Problem 4: Approximate Solution by HAM and HPM for $\alpha = \frac{1}{2}$



Problem 4: Approximate Solution by HAM and HPM for $\alpha = 1$

8. *Discussion of Results*

The results obtained for Problem 1 through HPM, and HAM are identical for both integral value of α and its fractional value. Meanwhile, the result presented in [1] via HAM was for integral value of α (i.e., $\alpha = 2$) only and it coincided with our result for the same value of α. In the present work, the results for both fractional and integral values of α are presented in 3D graphs. Problem 2 was solved in [1] by HAM and the result was presented for α but in addition to using HPM as a method of solution, results are presented for $\alpha = 1$ and $\alpha = \frac{1}{2}$ in the present work. Meanwhile, HAM and HPM have been used to reproduce results in [1]. The 3D graphs presented for the results for both methods show the closeness of the results. The results presented in the present work for Problem 3 by both HAM and HPM coincide with that in [4] where the method used is HPM. Here, we present results for fractional value of α and depict the results in 3D graphs for ease of comparison. Problem 4 was solved in [8] by HAM, while the present work applied HPM in addition to HAM and the results tally with those in [8]. Meanwhile, the 3D graph could not be drawn because the problem involves only one independent variable. No exact solution exists in the literature for Problem 5, with specific reference to [19] where the solution was presented for HAM and VIM. In the present work, we have applied both HPM and HAM to solve the same problem and our results compare well with those in [19]. 3D graphs are presented for the results.

Summary and Conclusion

- Solutions of fractional order partial differential equations of different categories by using both homotopy analysis method (HAM) and homotopy perturbation method (HPM) have been presented.
- The two semi analytic methods produce exact solution where such exists in closed form, and where such does not exist, the truncated series solution obtained give acceptable approximate result as can be seen in 3D graphs presented.
- In all cases, the accuracy of the results obtained via HPM despite the introduction of small parameter and the volume computations correspond with those obtained through HAM.

Future Direction

The algorithms presented in sections 4.2 and 5.2 shall be extended to multi fractional order partial differential equations and Sturm-Liouville problems.

Such attempts would be more involving in terms of computations, but, if it is eventually accomplished, it will be rewarding.

9. Appreciation

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