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Inventory management with hybrid cash-advance payment for time-dependent demand, time-varying holding cost and non-instantaneous deterioration under backordering and non-terminating situations

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Non-instantaneous decay

Abstract During the current pandemic period, the number of online transactions to purchase products through a mixed cash and prepayment agreement has increased significantly. Under this agreement, the buyer pays off a predefined percentage of the purchase price in advance, and the rest of the percentage is accomplished through the cash-on-delivery method. Adopting a practical scenario where the buyer has a storeroom with limited capacity, two sustainable inventory procedures for perishable items are developed, viz. (i) the inventory scheme with allowable shortages and fractional backlog, and (ii) the inventory scheme without ending. In both the schemes, not only the demand function but also the decay rate reveal an upward trend against storage time. However, the deterioration commencing moment is not instant but after a specific time interval from the products' storage moment. The cost of storing items is adopted as a linear function of storing time as the decay rate shows an ascending trend against storing time. A salient criterion is developed to examine when the non-ending situation with salvage values is more economical for the industry manager than the zero ending situation with the regular selling price. Finally, several decision-making findings are presented after completing four numerical examples along with a real-life case study for the validation of the inventory schemes.

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1. Introduction

Due to the concerns about the natural environment, the development of sustainable inventory management for perishable items has become a challenging research area as several deteriorated items (foods, vegetables, meat, dairy products, etc.) are vital sources of regular emissions into our environment. Inventory means usable resources of the products that can be used in the future through the products' utility durations. It is crucial to take care of the expected demand for future raw materials or furnished items. Manpower and machines can be used wisely to reduce the block of investment on them, but in absence of raw materials and furnished goods in their storages, the enterprise and the market system may collapse. However, keeping inventory indicates a blocking of investment without any profit. The cost of carrying, damage to the stock during storing and devaluation of the stock affect the profit mechanism of the enterprise significantly. Consequently, proper inventory management is a crying need of the hour so that the profit margin of the enterprise should not be affected.

Deterioration is an important factor in any inventory system that cannot be ignored in the analysis. This attribute of the products creates great interest among suppliers, manufactures, retailers as it brings the utility level of the goods down to zero due to decay, damage, vaporizations, or loss of value over time. Different kinds of food commodities like fruits, vegetables, etc. deteriorate due to direct spoilage that might be due to their perishable nature or insecurity in storing places from rats, cats, etc. Gasoline, alcohol, camphor, etc. deteriorate in the form of evaporation. Garments as well as electronic products gradually lose their value with the passage of time as new technology and fashion replace the old ones, reducing the value of the inventory. The practitioner's profit might be affected to a great extent because of the consequences of deterioration on stock, and hence, many researchers investigate its impacts by adopting different genres of decay rate based upon the nature of items. Ghare and Schrader [1] studied an inventory problem adopting a constant decay rate, while Covert and Philip [2] modified Ghare and Schrader [1]'s problem by adding a special type of two-parameter Weibull distributed decay. Shaikh et al. [3] designed an EOQ scheme for perishable items where decay commences after a certain length of time from the products' storing moment, while Jani et al. [4] investigated the consequences of deterioration on food items in India. Sarkar and Sarkar [5] further analysed the consequences of time varying decay under stock associated demand at each instant. Chowdhuri et al. [6] also conducted a thorough investigation into an inventory procedure incorporating a fixed percentage decay and demand that varies against price. Afterward, Shah et al. [7] found the best pricing policy for time varying decay products when demand is a multivariable function of price, trade period, and time. Shaikh et al. [8] formulated another inventory procedure exploiting a discount schedule in accordance with the purchase items for a constant decay product. Recently, Khan et al. [9–10] considered a variable decay rate depending on storage length and product lifespan.

When the decay rate increases over the storage duration, the cost of holding products is not always immutable rather than rising over the storage duration as well. The reason behind this is that the decision maker allots more money to prevent the number of deteriorated items and hence, he/she can reduce the amount of loss from the deteriorated items. Rahman et al. [11] relaxed the trivial consideration that the cost of carrying in unit time is constant by assuming a variable cost to store in unit time for each item in an interval environment, while Yang [12] constructed another inventory procedure assuming stock associated cost of holding products in warehouses instead of a constant cost. In the same year, realising the fact that the cost of holding non-decay items must be lower than that of holding decay items, Tyagi et al. [13] proposed an inventory-replenishing policy for delayed deteriorating goods. Recently, Dhandapani and Uthayakumar [14] studied another EOQ model for fresh fruits with preservation equipment along with a variable cost to carry the products over the storage period. Afterward, Shaikh et al. [8] and Khan et al. [15] explored the significance of the variable carrying cost on the profit for perishable items with a fixed decay percentage. Deterioration does not always take place as soon as the stocks are stored. For instance, fruits, vegetables, etc. remain fresh up to a certain duration from the storage moment and after that, decaying starts. This attribute is termed "delayed deterioration" or "non-instantaneous deterioration" in inventory studies. Tyagi et al. [13] further investigated the consequences of the delay deterioration on the best replenishment strategy when shortages appear. Tsao [16] established the combined effect of location, delayed deterioration, and preservation technology to retard the rate of decay on an inventory procedure. Chung et al. [17] adopted a discount strategy for deteriorating products under some criteria. Recently, Khan et al. [18] discussed various decay starting possibilities as a result of a better storing environment in a borrowed storage under a two-warehouse inventory procedure. All the aforementioned studies related to perishable items include either a constant/time varying instantaneous or constant non-instantaneous deterioration rate during the storage period. However, the deterioration of many perishable items (fruits, vegetables, cereals, seeds, etc.) is a non-instantaneous and increasing function with respect to the storage time as well. This gap in the management of perishable items is fulfilled in this study.

The demand for a product is directly related to its price and stock. An increase in price lessens the customers' demand and hence the inventory cycle increases. Similarly, if a substantial amount of stock is maintained, it will attract more customers. However, it is worth noting that it affects the profit margin as heavy investment is needed. An increase in stock also affects the inventory by increasing the number of deteriorated items. Sometimes, depending on the social status of the inhabitants of a locality, the low price of materials even decreases the demand, thereby putting an adverse effect on inventory. Sarkar [19] and Sett et al. [20] discussed both online and offline pricing strategies for a variable demand pattern. Bhuniya et al. [21] and Omair et al. [22] developed two distinct models in an uncertain environment where demand is fuzzy typed, whereas Avinadav et al. [23] investigated the combined effect of time and price on the demand for perishable items. Das et al. [24] worked on a production inventory problem incorporating price associate demand for perishable items. Prasad and Mukharjee [25] explored the effects of both stock and time dependent demand. Alfares and Ghaithan [26] derived a pricing scheme when the purchase price varies based upon the quantity amount, while Shah et al. [7] adopted preservation technology under selling price to study the inventory model.

Later, Khan et al. [9] improved the study of Alfares and Ghaithan [26] for decay items with a certain lifetime. Shaikh et al. [27] described two storing procedures for decaying items that integrate a ramp form demand. Pando et al. [28] and Cárdenas-Barrón et al. [29] investigated the best pricing policy for a practitioner reflecting customers' demand as the power form of price. Later on, Barman et al. [30] adopted a hybridization of market price and product greening level to reflect customers' demand appropriately. Recently, Rahman et al. [31] formulated an inventory problem hybridising price and current stock to reflect products' market demands. To attract more customers, the practitioners stock more products in the warehouse, which may create a non-ending situation in the business environment. To the authors' best knowledge, no study has been accomplished for non-instantaneous and time varying deteriorating items under a non-ending environment. Therefore, to help industry managers in order to manage non-instantaneous and time varying deteriorating items properly, it is necessary to develop a model under a non-ending situation.

Another vital part of the inventory model is "stock-out" scenario. Sometimes the retailer runs out of stock when there is still market demand. This is known as inventory backlogging, which refers to the delayed delivery of items to consumers. Consumers might wait for the late delivery, keeping in mind the goodwill of the retailer, or they might go to another place. Nevertheless, for the waiting customers, the retailer arranges the stock in spite of the loss from the nearby shop. Therefore, in this way, inventory backlogging has a negative aspect as it has a bad impact on inventory goodwill. Alshanbari et al. [32] derived an inventory model with waiting time dependent partial backlogging. Ahmed et al. [33] described partial backlogging with reworking opportunities for defective items. Sarkar and Sarker [5] formulated a model with time varying deterioration as well as stock dependent demand under incomplete backlog with a constant percentage. Prasad and Mukherjee [25] proposed a model by introducing variable demand into their model with deterioration and complete backlog, while Jaggi et al. [34] studied the effect of inflation on an EOQ model with fractional backlog. In this direction, Taleizadeh [35], Taleizadeh et al. [36], Khan et al. [37] and Rahman et al. [31] accomplished several inventory procedures by integrating some advanced features along with allowing shortages.

Advanced payment has an important role in the business world where buyers provide the entire or partial payment of the purchase price before receiving the product. In this situation, buyers create loans from other places, like bank or other sources, and hence, they need to have the capacity to pay the interest on loans. Shaikh et al. [38] incorporated advance payment coordination and designed an inventory problem solving with the application of particle swarm optimization (PSO). Next, Khan et al. [39] developed another inventory procedure with an advance payment method, considering the limited capacity of the retailer's warehouse. Duary et al. [40] presented a study for perishable items under an advance payment arrangement with trade credit and established optimal policies for the practitioner. Guria et al. [41] discussed a mixed prepayment and delay payment scheme along with a provision against the prepayment method, whereas Taleizadeh et al. [42] formulated an inventory procedure allowing fractional backlog under a fractional prepayment scheme. Similarly, Taleizadeh [43] proposed an identical multi-segment strategy instead of a single installment to accomplish the prepayment for vaporizing items allowing fractional backlog, while Zhang et al. [44] compared the consequences on the buyer's replenishing strategy between complete prepayment and mixed fractional-prepayment-fractional-delay payment schemes. Zia and Taleizadeh [45] further studied the mixed prepayment and delay payment methods defining a predefined threshold such that when the purchase product number is larger or equal to the threshold, the buyer is allowed to enjoy the hybrid scheme; otherwise, he/she has to follow the full prepayment method. Afterward, to make the supply chain coordination in stable, Zhang et al. [46] introduced prepayment arrangement in the two-phase supply chain and derived a best ordering policy for the system. Diabat et al. [47] improved Zia and Taleizadeh [45]'s study for decay items. Taleizadeh [35] evaluated advance payment pricing in an inventory coordination where shortages are backlogged partially with planning and then, proposed a suitable solution procedure to find the optimal solution. Again, Taleizadeh et al. [36] extended their previous work and formulated an inventory process with a prepay schedule where shortages are backlogged with planning. Later, Khan et al. [48] and Rahman et al. [31] incorporated a deduction opportunity of the purchase price in return of the prepayment. Recently, Khan et al. [37] proposed an order amount based mixed prepayment and delay payment schedule in such a way that the byer receives a higher portion for delay payment and a portion for prepayment against a higher order size. All the aforementioned studies related to advance payment schemes are completed with constant or stock or price sensitive demand patterns. However, the effects of advance payment for non-instantaneous and time varying deteriorating items under time dependent customer demand are not yet investigated. As a result, it becomes crucial to formulate a model incorporating the consequences of advance payment for non-instantaneous and time varying deteriorating items under time dependent customer demand.

In addition, the main contributions of the present study are presented in Table 1 so that anyone can straightforwardly recognize the novelty of the work.

As can be shown in Table 1, Mishra et al. [50] described the best replenishment scheme for a decay item when both demand and decay are related to time, but deterioration commences from the product storing moment, which is not compatible for many goods (for instance, meat, fruits, vegetables, cereals, seeds, and so on). Moreover, they adopted the fractional backlog with a fixed percentage and the capacity of the practitioner's storage as unlimited. However, in the practical world, sometimes the percentage of backordering is related with the arrival time duration of the new products, and hence, the backlog percentage related to the arrival time duration is much more compatible than the constant percentage. In addition, the storage capacity for a practitioner cannot be infinite in reality, and therefore, to reflect the business environment more appropriately, it is necessary to consider the limited capacity instead of the unlimited capacity. In reality, all the stored products are not always sold. That is, some products remain in the warehouse at the end of the inventory cycle and are sold using salvage value. However, this situation was not studied by Mishra et al. [50]. Finally, they included cash-on delivery as a payment method for the practitioner, but in today's competitive commercial environment, to reduce the

Table 1 Comparison of the study with related works in literature.

Literature	Payment scheme	Demand rate	Deterioration	Shortage	Holding cost	Storage capacity
Prasad and Mukherjee [25]	Cash	SD and time dependent	Time varying and IS	PB	Constant	Unlimited
Alfares and Ghaitan [26]	Cash	PD	No	No	Time varying	Unlimited
Shaikh et al. [27]	Cash	Ramp type	Constant and IS	PB	Constant	Unlimited
Pando et al. [28]	Cash	PD and SD	No	No	Constant	Unlimited
Cárdenas-Barrón et al. [29]	Cash	PD	No	No	Constant	Unlimited
Barman et al. [30]	Cash	Constant	No	No	Constant	Unlimited
Rahman et al. [31]	Advance	PD and SD	Constant and IS	PB	Constant	Unlimited
Alshanbari et al. [32]	Advance and cash	PD	Time varying and IS	PB	Constant	Unlimited
Admed et al. [33]	Cash	Constant	No	PB	Constant	Unlimited
Jaggi et al. [34]	Cash	Constant	Constant and IS	PB	Constant	Unlimited
Taleizadeh [35]	Advance	Constant	No	PB	Constant	Unlimited
Taleizadeh et al. [36]	Advance and delay	Constant	No	CB	Constant	Unlimited
Khan et al. [37]	Advance and delay	Constant	No	PB	Constant	Limited
Shaikh et al. [38]	Advance and cash	PD	Constant and IS	PB	Constant	Limited
Khan et al. [39]	Advance and cash	Constant	Constant and IS	PB	Constant	Limited
Duary et al. [40]	Advance and delay	Time dependent	Constant and IS	PB	Constant	Limited
Guria et al. [41]	Advance and delay	PD	No	No	Constant	Limited
Taleizadeh et al. [42]	Advance and cash	Constant	No	PB	Constant	Unlimited
Taleizadeh [43]	Advance	Constant	Constant and IS	PB	Constant	Unlimited
Zhang et al. [44]	Advance and delay	Constant	No	No	Constant	Unlimited
Zia and Taleizadeh [45]	Advance and delay	Constant	No	CB	Constant	Unlimited
Zhang et al. [46]	Advance	Constant	No	No	Constant	Unlimited
Diabat et al. [47]	Advance and delay	SD	Constant and IS	CB and PB	Constant	Unlimited
Khan et al. [48]	Advance	SD	Constant and IS	PB	Constant	Unlimited
Lashgari et al. [49]	Advance and delay	Constant	No	NS, CB and PB	Constant	Unlimited
Mishra et al. [50]	Cash	Time dependent	Time varying and IS	PB	Linear time dependent	Unlimited
Panda et al. [51]	Delay	Time dependent	Constant and IS	PB	Constant	Limited
This paper	**Mixed cash and advance**	**Time dependent**	**Time varying and NIS**	**PB and WE**	**Linearly time dependent**	**Limited**

PB: Partially backlogged, CB: Completely backlogged, NS: No shortages, WE: Without ending, IS: Instantaneous, NIS: Non-instantaneous, SD: Stock-dependent, PD: Price-dependent.

possibility of an uncertain cash flow, the supplier or manufacturer demands a certain percentage of the payment in advance from the retailer by offering on-time delivery assurance. Taking all these observations into account, this work improves the study of Mishra et al. [50] as follows:

i) Non-instantaneous decay is adopted instead of instantaneous decay.

ii) The backlogging percentage depends upon the arrival time duration being incorporated instead of a constant percentage.

iii) The storage capacity for a practitioner is considered as limited relaxing the unlimited capacity assumption.

iv) Without ending situation in the warehouse is inserted.

v) To reduce the possibility of uncertain cash flow, a mixed cash and prepayment agreement is adopted.

The combination of the aforementioned five features makes the current work unique in the inventory literature. Furthermore, the authors find a fish shop owner in Bangladesh who sells salmon fish after importing them from an overseas supplier, paying a fraction of the purchase price in advance. In addition, fishes commence to spoilage after a certain period and then, this deterioration rate increases over storage time. This real-life example motivates the authors to execute this research study. Fig. 1 exposes the relationships graphically among the members (supplier, retailer, and customers) of the entire system.

The organization of the remainder of the study is as follow. Section 2 mentions all the notations and basic considerations to articulate the inventory procedure, while the mathematical layout of the problem is delivered in the Section 3. Then, the Section 4 shows the solution process along with a flowchart. Four numerical problems along with a real-life case study are solved in Section 5. In Section 6, sensitivity analyses are accomplished for the fixed input factors of the problem in order to find several managerial insights. Finally, conclusions

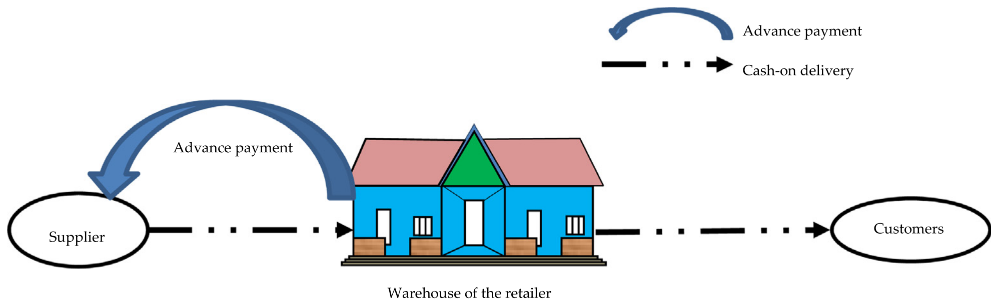

Fig. 1 Graphic illustration of the inventory system under advance payment.

of the study along with possible research scopes in future are highlighted in Section 7.

2. Notation and assumptions

To delineate the proposed inventory procedure mathematically the following notations and conventions are adopted in the paper.

Notations:

Notations	Description
A	Cost to place an order (\$/order)
a, b	Demand parameters $(a > 0, b > 0)$ (constant)
p	Market price (\$/unit)
c_p	Acquisition price (\$/unit)
$\boldsymbol{h}$	Constant cost to hold a unit item (\$/unit/time unit)
α	Time-varying cost to hold a unit item (\$/unit/(time unit)^2)
c_I	Opportunity cost (\$/unit)
c_s	Shortage cost (\$/unit)
$\psi(t)$	Time-varying decay
θ	Coefficient of time-varying decay $(0 < \theta < 1)$
t_d	Decay commencing time (time unit)
$\boldsymbol{R}$	Highest backlogged amount (units)
$\boldsymbol{S}$	Initial product amount in the storage (units)
p_s	Salvage price where $p_s < p$ (\$/units)
δ	Backlogging parameter
L	Allowed time duration to accomplish the prepayment
$\boldsymbol{n}$	Identical number of segment for the prepayment
β	Percentage of purchase price for the prepayment
U_w	Maximum capacity of the warehouse (units)
TC_i	Total cyclic cost for the inventor procedure $i = 1, 2$(\$/time unit)
Decision variables	
t_1	Duration of physical stock appearance in the storage (time unit) (for shortages case)
$\boldsymbol{B}$	Remaining number of products in without ending situation (units)
$\boldsymbol{T}$	Duration of a cycle (time unit)

Assumptions.

i. The decay rate $\psi(t)$ soars in the aftermath of time as $\psi(t) = \theta t$, where θ is a constant $(0 < \theta < 1)$, that is, this rate shows upward trend against storing time.

ii. There is no consequence of the deterioration on product amount during $[0, t_d]$. However, the number of products in the warehouse is affected by deterioration during $[t_d, t_1]$ with a variable deterioration rate $\psi(t)$.

iii. The demand rate of the product is linearly time and a unit vending price dependent as follows: $D = a - bp + ct,\ a, b > 0$ and $c \in R$(Panda et al. [51]). It is noteworthy that, demand reveals an ascending trend against time when the parameter c is positive, while demand reveals a descending trend against time when the parameter c is negative. Moreover, demand is independent from time when c is zero.

iv. The cost to hold a unit product for a unit time duration is proportional to the duration of the storage time of that unit, which is assumed as follows: $h(t) = h + \alpha t$, where $h, \alpha > 0$.

v. Shortages are permitted and when the stock out period starts, shortages are partly backlogged of the demand with the proportion $[1 + \delta(T - t)]^{-1}$, where $(T - t)$ is the arrival time length from any time $t \in [t_1, T]$ and $\delta > 0$.

vi. The inventory procedure is appropriate for a single non-instantaneous decay product.

vii. Neither replacement nor repair is permitted for deteriorated products.

viii. Inventory planning horizon and lead time are infinite and L respectively.

ix. The vendor, at the end of the cycle length for non-ending case, sells all the unsold inventory B units for salvages.

3. Problem definition

3.1. Inventory procedure when shortages perform

Suppose a vendor, initially, orders $(S + R)$ units of an item by paying β percentage of the total purchasing price to his supplier L months earlier amid n identical multiple segments at identical intervals during the allowed time L and pays the remaining $(1 - \beta)$ percentage at the receiving moment of the shipment. Then vendor, instantaneously, accomplishes all the backorder quantities R and therefore, the present number of quantities reduces to S. The amount S gradually decreases during $[0, t_d]$ because of the customers' demand $D(= a - bp + ct)$ only and after that deterioration starts i.e., stock decreases not only for customers' demand but also for time varying deterioration $\psi(t)$ during the time period $[t_d, t_1]$ and lastly, it depletes to zero at the time $t = t_1$. Subsequently, stock out situation is appeared and shortages are amassed during this situation based on the arrival time of the new shipment. This inventory system, based on the aforementioned assumptions, can be depicted in Fig. 2.

So the rate of change of the inventory, at $t \in [0, T]$, must satisfies the governing differential equations

$$\frac{dI_1(t)}{dt} = -(a - bp + ct),\ 0 \leqslant t \leqslant t_d, \tag{1}$$

$$\frac{dI_2(t)}{dt} + \psi(t)I_2(t) = -(a - bp + ct),\ t_d < t \leqslant t_1, \tag{2}$$

$$\frac{dI_3(t)}{dt} = -\frac{(a - bp + ct)}{1 + \delta(T - t)},\ t_1 < t \leqslant T, \tag{3}$$

with the supplementary conditions

$$I_1(t) = S \text{ at } t = 0, I_2(t) = 0 \text{ at } t = t_1 \text{ and } I_3(t) = -R \text{ at } t = T. \tag{4}$$

Moreover, $I(t)$ must hold the continuity at $t = t_d$ and $t = t_1$. On solving differential equations (1)-(3) with the boundary conditions (4), one has

$$I_1(t) = S - \left\{(a - bp)t + \frac{c}{2}t^2\right\},\ 0 \leqslant t \leqslant t_d, \tag{5}$$

$$I_2(t) = e^{-\frac{\theta t^2}{2}} \int_t^{t_1} e^{\frac{\theta u^2}{2}} (a - bp + cu)\, du, \quad t_d < t \leqslant t_1, \tag{6}$$

$$I_3(t) = \frac{1}{\delta}\left[\left(a - bp + c(\frac{1}{\delta} + T)\right)\ln|1 + \delta(T - t)| - c(T - t)\right] - R,\ \ t_1 \leqslant t \leqslant T. \tag{7}$$

Since the inventory level $I(t)$ holds the continuity at $t = t_d$ and $t = t_1$, so exploiting the continuity criterion at the points $t = t_d, t_1$ and expansion of exponential function, one can get easily the maximum inventory level, S, and maximum shortages, R, respectively.

$$\begin{aligned} S = (a - bp)&\left[\begin{array}{c} (t_1 - t_d) + \frac{\theta}{6}(t_1^3 - t_d^3) + \frac{\theta^2}{40}(t_1^5 - t_d^5) \\ -\frac{\theta}{2}(t_1 t_d^2 - t_d^3) - \frac{\theta^2}{12}(t_1^3 t_d^2 - t_d^5) + \frac{\theta^2}{8}(t_1 t_d^4 - t_d^5) \end{array}\right] \\ &+c\left[\begin{array}{c} \frac{1}{2}(t_1^2 - t_d^2) + \frac{\theta}{8}(t_1^4 - t_d^4) + \frac{\theta^2}{48}(t_1^6 - t_d^6) \\ -\frac{\theta}{4}(t_1^2 t_d^2 - t_d^4) - \frac{\theta^2}{16}(t_1^4 t_d^2 - t_d^6) + \frac{\theta^2}{16}(t_1^2 t_d^4 - t_d^6) \end{array}\right] \\ &+(a - bp)t_d + \frac{c}{2}t_d^2 \end{aligned} \tag{8}$$

and

$$R = \frac{1}{\delta}\left[\left(a - bp + c(\frac{1}{\delta} + T)\right)\ln|1 + \delta(T - t_1)| - c(T - t_1)\right] \tag{9}$$

So the total purchase cost including deterioration cost for each cycle is $PC = c_p(S + R)$.

As the inventory volume $I(t)$, in this inventory system, is always non-negative over $[0, t_1]$, consequently, the holding cost consists of two parts, namely, the cost to hold items during $[0, t_d]$ and the cost to hold items during $[t_d, t_1]$.

Now the holding cost in $[0, t_d]$ is

$$\int_0^{t_d} h(t)I_1(t)dt = \int_0^{t_d} (h + \alpha t)\left[S - \left\{(a - bp)t + \frac{c}{2}t^2\right\}\right] dt$$

$$= \left[\begin{array}{l} (a - bp)h\left[\begin{array}{c} \frac{1}{2}t_d^2 + (t_1 t_d - t_d^2) + \frac{\theta}{6}(t_d t_1^3 - t_d^4) + \frac{\theta^2}{40}(t_d t_1^5 - t_d^6) \\ -\frac{\theta}{2}(t_1 t_d^3 - t_d^4) - \frac{\theta^2}{12}(t_d^3 t_1^3 - t_d^6) + \frac{\theta^2}{8}(t_1 t_d^5 - t_d^6) \end{array}\right] \\ +ch\left[\begin{array}{c} \frac{1}{2}(t_1^2 t_d - t_d^3) + \frac{\theta}{8}(t_1^4 t_d - t_d^5) - \frac{\theta^2}{48}(t_1^6 t_d - t_d^7) - \frac{\theta}{4}(t_1^2 t_d^3 - t_d^5) \\ -\frac{\theta^2}{16}(t_1^4 t_d^3 - t_d^7) + \frac{\theta^2}{16}(t_1^2 t_d^5 - t_d^7) + \frac{t_d^3}{3} \end{array}\right] \\ +\alpha(a - bp)\left[\begin{array}{c} \frac{1}{2}(t_1 t_d^2 - t_d^3) + \frac{\theta}{12}(t_1^3 t_d^2 - t_d^5) + \frac{\theta^2}{80}(t_1^5 t_d^2 - t_d^7) \\ -\frac{\theta}{4}(t_1 t_d^4 - t_d^5) - \frac{\theta^2}{24}(t_1^3 t_d^4 - t_d^7) + \frac{\theta^2}{16}(t_1 t_d^6 - t_d^7) + \frac{t_d^3}{6} \end{array}\right] \\ +c\alpha\left[\begin{array}{c} \frac{1}{4}(t_1^2 t_d^2 - t_d^4) + \frac{\theta}{16}(t_1^4 t_d^2 - t_d^6) - \frac{\theta^2}{96}(t_1^6 t_d^2 - t_d^8) \\ -\frac{\theta}{8}(t_1^2 t_d^4 - t_d^6) - \frac{\theta^2}{32}(t_1^4 t_d^4 - t_d^8) + \frac{\theta^2}{32}(t_1^2 t_d^6 - t_d^8) + \frac{t_d^4}{8} \end{array}\right] \end{array}\right]$$

Again, the holding cost in $[t_d, t_1]$, exploiting expansion of exponential function, is

$$\int_{t_d}^{t_1} h(t)I_2(t)dt = \int_{t_d}^{t_1} (h + \alpha t)e^{-\frac{\theta t^2}{2}} \int_t^{t_1} e^{\frac{\theta u^2}{2}}(a - bp + cu)\, du\, dt$$

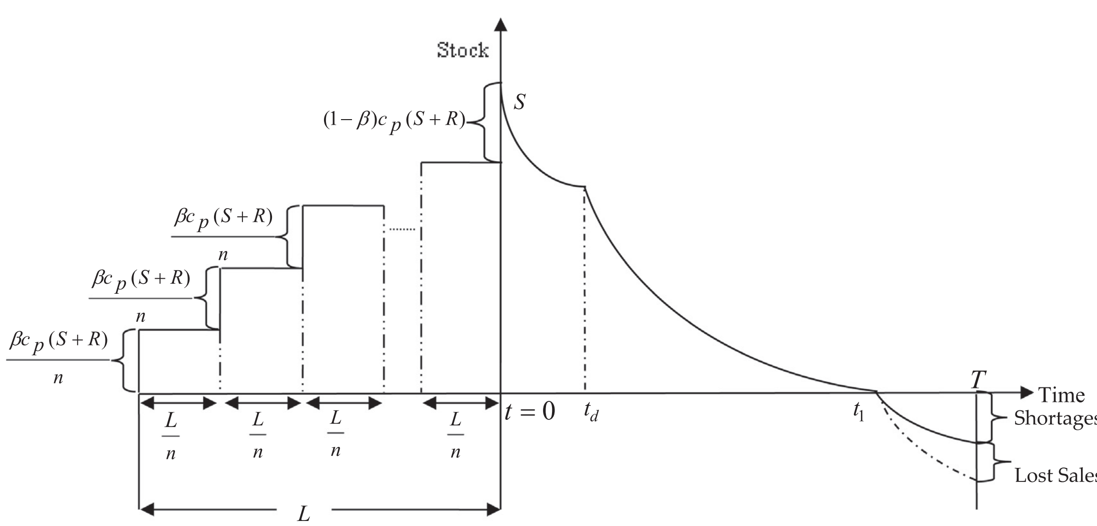

Fig. 2 Inventory procedure with the hybrid cash-prepayment approach when shortages perform.

$$= \begin{bmatrix} (a-bp)h\left\{\frac{1}{2}(t_1^2-t_d^2)+\frac{\theta}{12}(t_1^4-t_d^4)+\frac{\theta^2}{90}(t_1^6-t_d^6)+\frac{\theta^3}{840}(t_1^8-t_d^8)\right\} \\ +ch\left\{\frac{1}{3}(t_1^3-t_d^3)+\frac{\theta}{15}(t_1^5-t_d^5)+\frac{\theta^2}{105}(t_1^7-t_d^7)+\frac{\theta^3}{945}(t_1^9-t_d^9)\right\} \\ +(a-bp)\alpha\left\{\frac{1}{6}(t_1^3-t_d^3)+\frac{\theta}{40}(t_1^5-t_d^5)+\frac{\theta^2}{336}(t_1^7-t_d^7)+\frac{\theta^3}{3456}(t_1^9-t_d^9)\right\} \\ +c\alpha\left\{\frac{1}{8}(t_1^4-t_d^4)+\frac{\theta}{48}(t_1^6-t_d^6)+\frac{\theta^2}{384}(t_1^8-t_d^8)+\frac{\theta^3}{3840}(t_1^{10}-t_d^{10})\right\} \end{bmatrix}$$

Therefore, the entire cost to hold all items for a single replenishment is

$$HC = \int_0^{t_d} h(t)I_1(t)dt + \int_{t_d}^{t_1} h(t)I_2(t)dt$$

As the stock level becomes zero at $t = t_1$, shortly after stock out situation is appeared and shortages are amassed under this situation based on the arrival time of the new shipment during $[t_1, T]$. So the cost for the stock out circumstances for a single replenishment is

$$SC = -c_s\int_{t_1}^{T} I_3(t)\,dt$$

$$= c_s\left[R(T-t_1) - \frac{1}{\delta}\left[\begin{array}{c}(a-bp+c(\frac{1}{\delta}+T)) \\ \left(\frac{1}{\delta}\{1+\delta(T-t_1)\}\ln|1+\delta(T-t_1)| - (T-t_1)\right) - \frac{c}{2}(T-t_1)^2\end{array}\right]\right]$$

Since shortages are not amassed completely, as a result, the practitioner faces some losses of sale and the resultant lost sale cost is

$$LSC = c_l\int_{t_1}^{T} D\left(1-\frac{1}{1+\delta(T-t)}\right)dt$$

$$= c_l\left[\int_{t_1}^{T}(a-bp+ct)\,dt - \int_{t_1}^{T}\frac{(a-bp+ct)}{1+\delta(T-t)}\,dt\right]$$

$$= c_l\left[\begin{array}{c}(a-bp)(T-t_1)+\frac{c}{2}(T^2-t_1^2) \\ -\frac{1}{\delta}\left(a-bp+c(\frac{1}{\delta}+T)\right)\ln|1+\delta(T-t_1)| + \frac{c}{\delta}(T-t_1)\end{array}\right]$$

The cost for the prepayment amount for a single replenishment is computed straightforwardly from Fig. 2. Thus, the capital cost is

$$I_c\left[\frac{\beta c_p(S+R)}{n}.\frac{L}{n}(1+2+3+\ldots+n)\right] = \frac{n+1}{2n}I_cL\beta c_p(S+R)$$

Now the practitioner's total cost, during the completion of the entire cycle, is composed of the sum of the ordering cost (A), the purchase cost (PC), the holding cost (HC), the shortage cost (SC), the lost sale cost (LSC) and the capital cost. Hence, the total cyclic cost is

$$A + \left(1+\frac{n+1}{2n}I_cL\beta\right)c_p(S+R) + HC + SC + LSC$$

Finally, the practitioner's cost for a unit time is

$$TC_1(t_1,T) = \frac{1}{T}\left[A+\left(1+\frac{n+1}{2n}I_cL\beta\right)c_p(S+R)+HC+SC+LSC\right] \quad (10)$$

Now the utmost target is to achieve the best values of t_1 and T (say, t_1^* and T^*) to optimize the practitioner's cost $TC_1(t_1, T)$.

3.2. Non-ending inventory procedure

In this case, the vendor receives the order of fresh S units at $t = 0$ after paying the total purchasing cost proceeding on the similar way to case 3.1. This inventory level S decreases during $[0, t_d]$ because of the customers' demand $D(= a - bp + ct)$ only and after that deterioration starts exactly at $t = t_d$. Then stock level diminishes with the joint effects of customers' demand and decay during $[t_d, T]$ and finally, it depletes to B at $t = T$. Towards the end of the cycle, vendor sells these B units for salvage values before receiving the next order of fresh S units. This inventory system can be depicted in Fig. 3.

Therefore, the number of items in the stock at $t \in [0, T]$ can be delineated as follows

$$\frac{dI_1(t)}{dt} = -(a-bp+ct), \quad 0 \leqslant t \leqslant t_d, \quad (11)$$

$$\frac{dI_2(t)}{dt} + \psi(t)I_2(t) = -(a-bp+ct), \quad t_d \leqslant t \leqslant T, \quad (12)$$

subject to the auxiliary conditions

$$I_1(t) = S \text{ at } t = 0 \text{ and } I_2(t) = B \text{ at } t = T. \quad (13)$$

In addition, $I(t)$ must hold the continuity at $t = t_d$. By solving differential equations (11) and (12) with the boundary conditions (13) and the expansion of exponential function, one can get

$$I_1(t) = S - \left\{(a-bp)t + \frac{c}{2}t^2\right\} \; 0 \leqslant t \leqslant t_d \quad (14)$$

$$I_2(t) = \Delta\left(1-\frac{\theta}{2}t^2+\frac{\theta^2}{8}t^4-\frac{\theta^3}{48}t^6\right) - (a-bp)\left(t-\frac{\theta}{3}t^3+\frac{\theta^2}{15}t^5\right) \; t_d \leqslant t \leqslant T$$
$$-c\left(\frac{1}{2}t^2-\frac{\theta}{8}t^4+\frac{\theta^2}{48}t^6\right) \quad (15)$$

where

$$\Delta = Be^{\frac{\theta T^2}{2}} + (a-bp)\left(T+\frac{\theta}{6}T^3+\frac{\theta^2}{40}T^5+\frac{\theta^3}{336}T^7\right) + c\left(\frac{1}{2}T^2+\frac{\theta}{8}T^4+\frac{\theta^2}{48}T^6+\frac{\theta^3}{384}T^8\right).$$

Since the inventory level $I(t)$ holds the continuity at $t = t_d$, so exploiting the continuity criterion at the point $t = t_d$, the maximum inventory level S can be expressed as follows

$$S = \Delta\left(1-\frac{\theta}{2}t_d^2+\frac{\theta^2}{8}t_d^4-\frac{\theta^3}{48}t_d^6\right) + (a-bp)\left(\frac{\theta}{3}t_d^3-\frac{\theta^2}{15}t_d^5\right) + c\left(\frac{\theta}{8}t_d^4-\frac{\theta^2}{48}t_d^6\right) \leqslant U_w \quad (16)$$

Therefore, the total purchase cost including deterioration cost for a single cycle is

$$PC = c_pS \quad (17)$$

Since the inventory volume $I(t)$ under this inventory procedure, is always non-negative over $[0, T]$, subsequently, the holding cost consists of two parts, namely, the cost to hold items during $[0, t_d]$ and the cost to hold items during $[t_d, T]$.

Now the holding cost in $[0, t_d]$ is

$$\int_0^{t_d} h(t)I_1(t)dt = \int_0^{t_d}(h+\alpha t)\left[S-\left\{(a-bp)t+\frac{c}{2}t^2\right\}\right]\,dt$$

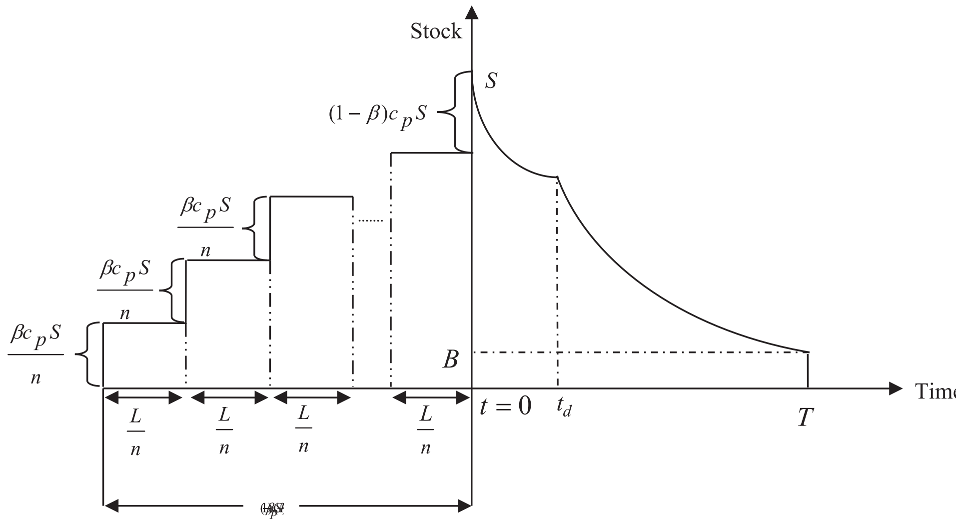

Fig. 3 Inventory procedure with the hybrid cash-prepayment approach when non-terminating situation appears.

$$= \left(ht_d + \frac{1}{2}\alpha t_d^2\right)S - h\left[\frac{1}{2}(a-bp)t_d^2 + \frac{1}{6}ct_d^3\right] - \alpha\left[\frac{1}{3}(a-bp)t_d^3 + \frac{1}{8}ct_d^4\right]$$

Again, the holding cost in $[t_d, T]$, exploiting expansion of exponential function, is

$$\int_{t_d}^{T} h(t)I_2(t)dt = \int_{t_d}^{T}(h+\alpha t)I_2(t)dt$$

$$\begin{aligned} &= \Delta h\left\{(T-t_d) - \tfrac{\theta}{6}(T^3-t_d^3) + \tfrac{\theta^2}{40}(T^5-t_d^5) - \tfrac{\theta^3}{336}(T^7-t_d^7)\right\} \\ &+\Delta\alpha\left\{\tfrac{1}{2}(T^2-t_d^2) - \tfrac{\theta}{8}(T^4-t_d^4) + \tfrac{\theta^2}{48}(T^6-t_d^6) - \tfrac{\theta^3}{384}(T^8-t_d^8)\right\} \\ &-(a-bp)h\left\{\tfrac{1}{2}(T^2-t_d^2) - \tfrac{\theta}{12}(T^4-t_d^4) + \tfrac{\theta^2}{90}(T^6-t_d^6)\right\} \\ &-ch\left\{\tfrac{1}{6}(T^3-t_d^3) - \tfrac{\theta}{40}(T^5-t_d^5) + \tfrac{\theta^2}{336}(T^7-t_d^7)\right\} \\ &-(a-bp)\alpha\left\{\tfrac{1}{3}(T^3-t_d^3) - \tfrac{\theta}{15}(T^5-t_d^5) + \tfrac{\theta^2}{105}(T^7-t_d^7)\right\} \\ &-c\alpha\left\{\tfrac{1}{8}(T^4-t_d^4) - \tfrac{\theta}{48}(T^6-t_d^6) + \tfrac{\theta^2}{384}(T^8-t_d^8)\right\} \end{aligned}$$

Therefore, the total holding cost for a unique replenishment is

$$HC = \int_0^{t_d} h(t)I_1(t)dt + \int_{t_d}^{t_1} h(t)I_2(t)dt \tag{18}$$

In the last moment of the cycle i.e., at $t = T$, the vendor sells all the ending inventory with the salvage price p_s per unit. Therefore, the salvage value is

$$SV = p_s B$$

The cost for the prepayment amount for a single replenishment is computed straightforwardly from Fig. 3. Thus, the capital cost is

$$I_c\left[\frac{\beta c_p S}{n}.\frac{L}{n}(1+2+3+\ldots+n)\right] = \frac{n+1}{2n}I_c L\beta c_p S \tag{19}$$

Therefore, the practitioner's total cost, during the completion of the entire cycle, is

$$A + \left(1+\frac{n+1}{2n}I_c L\beta\right)c_p S + HC - SV$$

Finally, the practitioner's cyclic cost for a unit time is

$$TC_2(B,T) = \frac{1}{T}\left[A + \left(1+\frac{n+1}{2n}I_c L\beta\right)c_p S + HC - SV\right] \tag{20}$$

Now the ultimate goal is to reach the best values of B and T (say, B^* and T^*) to optimize the practitioner's cost $TC_2(B,T)$.

4. Solution process

In the present section, the computational solution technique for both aforementioned inventory procedures is deliberated. The solution process for the inventory scheme with backordering is delineated at first and secondly, that of for the inventory scheme for without ending situation.

4.1. Inventory procedure when shortages perform

The present section provides the necessary and sufficient conditions for optimizing the total inventory cost of the inventory scheme with backordering.

The optimal values of t_1 and T(say t_1^* and T^*) to minimize the total cyclic cost for a unit time $TC_1(t_1,T)$ can be found after solving the following necessary conditions simultaneously:

$$\frac{\partial TC_1}{\partial t_1} = 0 \text{ and } \frac{\partial TC_1}{\partial T} = 0 \tag{21}$$

provided that they must satisfy the subsequent sufficient conditions

$$\left.\frac{\partial^2 TC_1}{\partial t_1^2}\right|_{(t_1^*,T^*)} < 0,\ \left.\frac{\partial^2 TC_1}{\partial T^2}\right|_{(t_1^*,T^*)} < 0 \text{ and } \left.\left\{\frac{\partial^2 TC_1}{\partial t_1^2}\frac{\partial^2 TC_1}{\partial T^2} - \left(\frac{\partial^2 TC_1}{\partial t_1 \partial T}\right)^2\right\}\right|_{(t_1^*,T^*)} < 0$$

As our objective function $TC_1(t_1, T)$ is highly nonlinear, the necessary criterions in equation (21) are solved by the eminent generalized reduced gradient (GRG) method to achieve the best replenishment strategy to optimize the cost.

4.2. Non-ending inventory procedure

The ending inventory B, from equation (16), must satisfy the following inequality

$$B \leqslant e^{-\frac{\theta T^2}{2}}\left[\frac{1}{\xi_1}(U_w - \varepsilon_1 - \varepsilon_2) - \varepsilon_3 - \varepsilon_4\right] \tag{22}$$

where,

$$\varepsilon_1 = (a - bp)\left(\frac{\theta}{3}t_d^3 - \frac{\theta^2}{15}t_d^5\right), \varepsilon_2 = c\left(\frac{\theta}{8}t_d^4 - \frac{\theta^2}{48}t_d^6\right)$$

$$\varepsilon_3 = (a - bp)\left(T + \frac{\theta}{6}T^3 + \frac{\theta^2}{40}T^5 + \frac{\theta^3}{336}T^7\right)$$

$$\varepsilon_4 = c\left(\frac{1}{2}T^2 + \frac{\theta}{8}T^4 + \frac{\theta^2}{48}T^6 + \frac{\theta^3}{384}T^8\right),$$

$$\xi_1 = \left(1 - \frac{\theta}{2}t_d^2 + \frac{\theta^2}{8}t_d^4 - \frac{\theta^3}{48}t_d^6\right)$$

To optimize the cost $TC_2(T, B)$, calculate the first order and then, the second order derivatives (partial) of $TC_2(T, B)$ against B for any fixed value of T, one get

$$\frac{\partial TC_2}{\partial B} = \frac{1}{T}\left[\left\{\left(1 + \frac{n+1}{2n}I_c L\beta\right)c_p\xi_1 + \xi_1\varepsilon_5 + h\varepsilon_6 + \alpha\varepsilon_7\right\}e^{\frac{\theta T^2}{2}} - p_s\right] \tag{23}$$

where,

$$\varepsilon_5 = \left(ht_d + \frac{1}{2}\alpha t_d^2\right)$$

$$\varepsilon_6 = \left\{(T - t_d) - \frac{\theta}{6}(T^3 - t_d^3) + \frac{\theta^2}{40}(T^5 - t_d^5) - \frac{\theta^3}{336}(T^7 - t_d^7)\right\}$$

$$\varepsilon_7 = \left\{\frac{1}{2}(T^2 - t_d^2) - \frac{\theta}{8}(T^4 - t_d^4) + \frac{\theta^2}{48}(T^6 - t_d^6) - \frac{\theta^3}{384}(T^8 - t_d^8)\right\}$$

and

$$\frac{\partial^2 TC_2}{\partial B^2} = 0 \tag{24}$$

Equation (24) reveals that the expression of $\frac{\partial TC_2}{\partial B}$ is independent from B. Consequently, whether the procedure involves without ending scenario is exactly depends upon the value of $\frac{\partial TC_2}{\partial B}$.

For convenience, let us define

$$\Omega = \left\{\left(1 + \frac{n+1}{2n}I_c L\beta\right)c_p\xi_1 + \xi_1\varepsilon_5 + h\varepsilon_6 + \alpha\varepsilon_7\right\}e^{\frac{\theta T^2}{2}} - p_s \tag{25}$$

Now an important criterion is constructed in the next theorem to decide whether the inventory procedure involves without ending situation to minimize the practitioner cost.

Theorem. Let Ω *be defined by equation (25). Then,*

(a) *If* $\Omega \geqslant 0$, *then* $TC_2(T, B)$ *is minimized at* $B^* = 0$.
(b) *If* $\Omega < 0$, *then* $TC_2(T, B)$ *is minimized at* $B^* = e^{-\frac{\theta T^2}{2}}\left[\frac{1}{\xi_1}(U_w - \varepsilon_1 - \varepsilon_2) - \varepsilon_3 - \varepsilon_4\right]$.

Proof.

(a) If $\Omega > 0$, then the total cyclic cost $TC_2(T, B)$ is always an increasing function of the ending inventory level B. Since $TC_2(T, B)$ is a linear function in B, so from (22) one has $0 \leqslant B \leqslant e^{-\frac{\theta T^2}{2}}\left[\frac{1}{\xi_1}(U_w - \varepsilon_1 - \varepsilon_2) - \varepsilon_3 - \varepsilon_4\right]$ and consequently, $TC_2(T, B)$ will be minimized at $B^* = 0$. Moreover, when $\Omega = 0$, then the cost $TC_2(T, B)$ must be constant with respect to B, that is, $TC_2(T, B)$ does not involve B any more. As a result, $TC_2(T, B)$ will be minimized at $B^* = 0$. This ends the proof of the part (a).

(b) Again, if $\Omega > 0$, then the total cyclic cost $TC_2(T, B)$ is always a decreasing function of the ending inventory level B and hence, the best ending inventory level can be found at the upper boundary point of B. Therefore, $TC_2(T, B)$ will be minimized at $B^* = e^{-\frac{\theta T^2}{2}}\left[\frac{1}{\xi_1}(U_w - \varepsilon_1 - \varepsilon_2) - \varepsilon_3 - \varepsilon_4\right]$.

Therefore, there are two possibilities for the optimal value of B. If we substitute these optimal values B^* (say $B_1^*(=0)$ and $B_2^*\left(= e^{-\frac{\theta T^2}{2}}\left[\frac{1}{\xi_1}(U_w - \varepsilon_1 - \varepsilon_2) - \varepsilon_3 - \varepsilon_4\right]\right)$), from the above theorem, separately in Eq. (20), then we will get two highly non-linear function of single variable T only, namely, $TC_{2,1}(T)$ and $TC_{2,2}(T, B_2^*)$. Then, the best replenishment length can be achieved by solving

$$\frac{dTC_{2,1}(T)}{dT} = 0 \text{ and } \frac{dTC_{2,2}(T, B_2^*)}{dT} = 0 \tag{26}$$

Since these two equations are highly nonlinear, the closed form values for T are not possible to achieve. Consequently, anyone can solve these by either the eminent generalized reduced gradient (GRG) method or classical numerical method for the optimal values of T(say T_1^* and T_2^*).

Utilizing the above outcomes, a set of steps is constructed to attain the best inventory strategy for the practitioner. Furthermore, to accentuate the implicit functionality of the proposed set of steps, a flowchart is provided in Fig. 4.

Algorithm 1 (*Computational steps to obtain the optimal solution*).
Step 1. Insert the input parameters: $A, a, b, c, p, p_s, c_p, h, \alpha, t_d, \theta, L, I_c, n, \beta$ and U_w.

Step 2. Initialize $T = t_d$ and $TC_2^{(\min)}(B, T) = \infty$.

Step 3. Compute Ω. If $\Omega \geqslant 0$, put $B^* = 0$ and toward the Step 4; else, toward the Step 5.

Step 4. Solve $\frac{dTC_2(B=0,T)}{dT} = 0$ for T^*. Calculate $TC_2(B^*, T^*)$ and toward the Step 6.

Step 5. Set $B^* = e^{-\frac{\theta T^2}{2}}\left[\frac{1}{\xi_1}(U_w - \varepsilon_1 - \varepsilon_2) - \varepsilon_3 - \varepsilon_4\right]$ and solve $\frac{dTC_2(B^*,T)}{dT} = 0$ for T^*. Calculate $TC_2(B^*, T^*)$.

Step 6. If $TC_2(B^*, T^*) < TC_2^{(\min)}(B, T)$, then update the values of $TC_2^{(\min)}(B, T)$ by $TC_2(B^*, T^*)$,T by T^* and toward the Step 3. Else, toward the Step 7.

Step 7. Report the best strategy for the practitioner: B^*,T^* and $TC_2^{(\min)}(B^*, T^*)$

Step 8. Stop.

5. Results and discussion

To examine the presented inventory procedures numerically, in this section, a real case study and four different numerical examples are supplied.

5.1. Case study

A fish shop owner in Bangladesh sells imported salmon fishes after purchasing from a foreign supplier. The foreign supplier

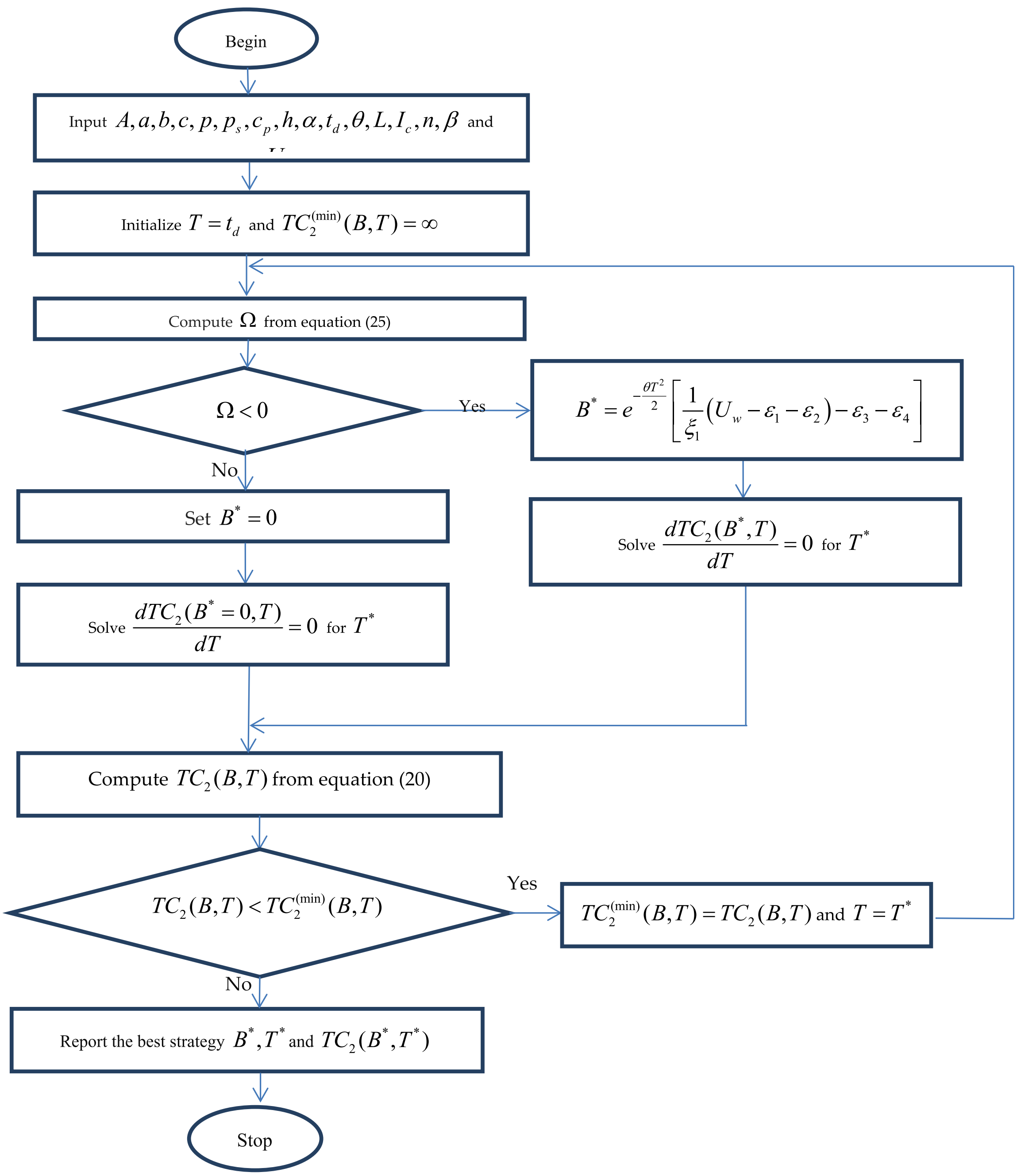

Fig. 4 Flowchart of the solution process for non-ending inventory scheme.

imposes a restriction to the fish shop owner that the shop owner has to pay 40% of the purchase cost through three equal installments at equal 15 days interval prior to the delivery. The rate of interest for the advance payments is 4.45% in a year. The shop owner stores the imported fishes in his warehouse with a cold storage facility having a variable charge of $4 for each container in a day and the owner allots extra $0.5 to keep the remaining fish containers for each day to keep the fishes' original quality for a longer period. Fishes are sold from the shop after bringing from the cold storage in every morning. The ordering cost for the shop owner includes the delivery charges $1000 per delivery and customs charge of $200 per delivery.

The customers' monthly demand is 24 cartons salmon fishes; customers' annual demand decreases ¼ carton against a unit selling price increment. Moreover, the selling price and the purchase cost for each carton are $300 and $200 respectively. Previous data reveal that the estimated deterioration free duration is 950 h and the coefficient of the time varying deterioration rate is 0.1. Before starting the deterioration, the customers demand increases 2 cartons salmon fishes per day while it decreases 1.5 cartons per day after starting the spoilage of salmon fishes. When all the stored fishes are sold, shortages appear. The estimated backlogging parameter related with the waiting time based on the experience is 1.5. The shortage and lost opportunity costs for the shop owner are $210 and $250 for every carton in a year respectively. Consequently, the shop owner's inventory data in notation are as follows: $A = \$(1000 + 200)/\text{order} = \$1200/\text{order}$, $a = (24 \times 12) = 288$ units, $b = 0.25$, $c = \begin{cases} 24 \text{ units/year}, & \text{when deteriration does not appear} \\ -18 \text{ units/year}, & \text{when deteriration appears} \end{cases}$, $c_p = \$200/\text{unit}$, $p = \$300/\text{unit}$, $c_s = \$210$ /unit/year, $c_l = \$250$ / unit/year $h = \$(4 \times 365)/\text{unit/year} = \$1460/\text{unit/year}$, $\alpha = 182.5/\text{unit}/(\text{year})\hat{}2$, $t_d = \frac{950}{8760}$ years ,$L = \frac{46}{365}$ year, $\theta = 0.1$, $I_c = 4.45\%/\text{year}$, $n = 3$, $\beta = 40\%$ and $\delta = 1.5$.

In this case, the optimal inventory policy for the fish shop owner utilizing LINGO 14.0 are:

$t_1^* = 0.1084$ year, $T^* = 0.4579$ year, ordering quantity is 81.5967 cartons fishes and $TC^* = 54586.39$/year (see Fig. 5). In addition, Fig. 5 reveals the joint convexity of the objective function with respect to the decision variables.

5.2. Numerical examples

Example 1. *In order to observe the enhancement of Mishra et al. [50] numerically, the numerical problem in Mishra et al. [50] is considered and solved after inserting some additional date to adjust with the formulated model. The inventory date are: $A = \$2500/order$, $a = 10$ units/month, $b = 0.25$, $c = 50$, $p = \$30/unit$, $c_p = \$10/unit$, $c_s = \$4/unit/month$, $c_l = \$8/unit/month$ $h = \$0.5$ / unit/month, $\alpha = 20/unit/(month)\hat{}2$, $t_d = 0.1$ month, $\theta = 0.8$, $L = 0.5$ month, $I_c = \$0.01/month$, $n = 10$, $\beta = 0.6$ and $\delta = 0.04$. Using LINGO optimization package version 14, the optimum inventory policy is obtained as $t_1^* = 0.6429$ month, $T^* = 2.5018$ months and $TC^* = 1837.633$ (see Fig. 6). In addition, one can investigate the joint convexity of the objective function with respect to the decision variables from Fig. 6 straightforwardly.*

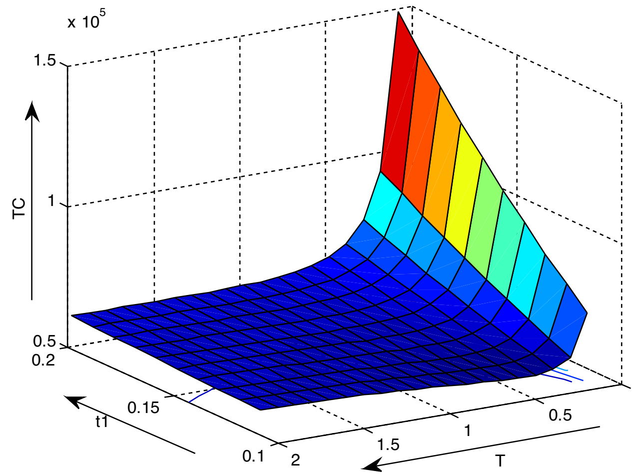

Fig. 5 Graphical observation of the annual total cost of the fish shop owner.

Fig. 6 exposes the convexity of the cyclic cost function graphically with respect to the decision variables. Now, the derived optimal numerical values are compared with those in Mishra et al. [50]. These are displayed in the following Table 2:

Based on the numerical comparison Table 2, the following salient remarks are highlighted. First of all, the optimal cycle period (T^*) from the present study is larger than the optimal T^* from Mishra et al. [50]. Since a negative impact of the selling price in the customers' demand is incorporated for the deteriorating items, the considered demand is always smaller than the demand in Mishra et al. [50] and hence, the optimal replenishment length is longer than that of from Mishra et al. [50]. However, the optimal stock-in duration (t_1^*) is smaller than the optimal t_1^* from Mishra et al. [50]. The reason behind of this attribute is again the lower demand pattern in this work. Since the demand rate is smaller and the products' carrying cost is linearly increasing with respect to the storage period, the practitioner makes a smaller lot size to avoid a large holding cost. Consequently, all the purchased quantities being sold within a smaller period. Finally, from the aspect of

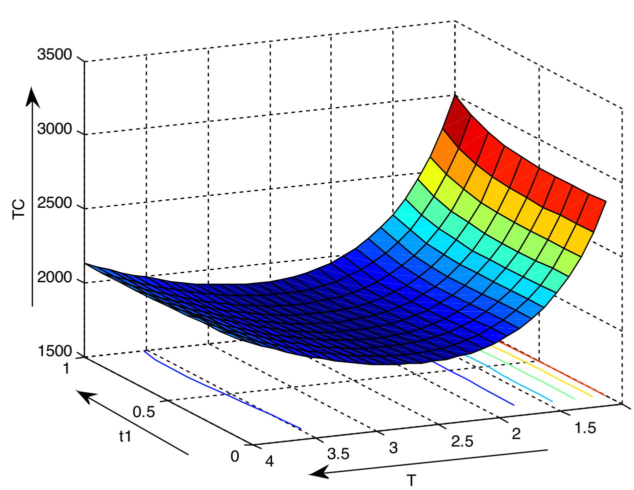

Fig. 6 Graphical observation of the cyclic cost against the decision variables.

Table 2 Numerical comparison of the derived results with Mishra et al. [50].

Optimal results from the present study	Optimal results from Mishra et al. [50]
$t_1^* = 0.6429$ month, $T^* = 2.5018$ months and $TC^* = 1837.633$	$t_1^* = 1.127$ month, $T^* = 1.562$ months and $TC^* = 2463.65$

cyclic total cost per unit time, the present study provides a better result than Mishra et al. [50]. In the case of Mishra et al. [50], the purchased quantities are stored in the warehouse for a longer period and the products' carrying cost is linearly increasing with respect to storage time; as a result, the total holding cost in Mishra et al. [50] is significantly larger than the present study. Again, a constant backlogging rate (80%) is considered in Mishra et al. [50] while a waiting time dependent backlogging rate is adopted in this study. Therefore, both the shortage and lost opportunity costs in this work are smaller than that of from Mishra et al. [50]. Moreover, in Mishra et al. [50], the holding cost and maximum inventory level are computed taking the expansion of exponential functions up to the second term; however, in the present study, expansions are taken into account up to third term during the computation of both holding cost and maximum stock in the warehouse. Consequently, the derived optimal solution from this paper is more accurate and economical compared with Mishra et al. [50].

Example 2. *Inventory procedure when shortages perform*

To examine the applicability of the formulated model suitably, let us consider the inventory procedure changing the values of A, a, b, c, θ, c_p, c_s, c_l and h in Example 1 (from Mishra et al. [50]) as follows :$A = \$1000$/order, $a = 250$ units/month, $b = 0.5$, $c = 4$, $c_p = \$20$/unit, $c_s = \$10$ /unit/month, $c_l = \$15$ /unit/month $h = \$0.25$ /unit/month and $\theta = 0.03$.

We use LINGO 14.0 for this example and find the following optimum solution:

$t_1^* = 0.6987$ months (or 21 days), $T^* = 1.2566$ months (38 days), $S^* = 165.565$ units, $R^* = 131.818$ units and $TC^* = 6093.712$(see Fig. 7). In addition, Fig. 4 reveals the joint convexity of the practitioner's cyclic cost against t_1 and T.

Example 3. *Non-ending inventory procedure*

To explore the applicability of the developed non-ending inventory procedure, let us consider another inventory procedure changing only deterioration free period t_d by 0.8 month in Example 2. Furthermore, the maximum storage and salvage price are $U_w = 300$ and $p_s = \$20$/unit respectively. LINGO 14.0 optimization tool is used for this example and the following optimum solution is found:

$B^* = 110.72$ units, $S^* = 300$ units, $T^* = 0.8$ months (24 days) and $TC^* = 6469.181$ (see Fig. 8). Moreover, the value of Ω as per our considered example is – 0.367 , which is negative and apparently, in accordance to the part (b) of the constructed Theorem, there must exist some unsold products in the optimal inventory procedure and these are salvaged. Furthermore, the joint convexity of the practitioner's cyclic cost with respect to the decision variables t_1 and T is exposed graphically in Fig. 8.

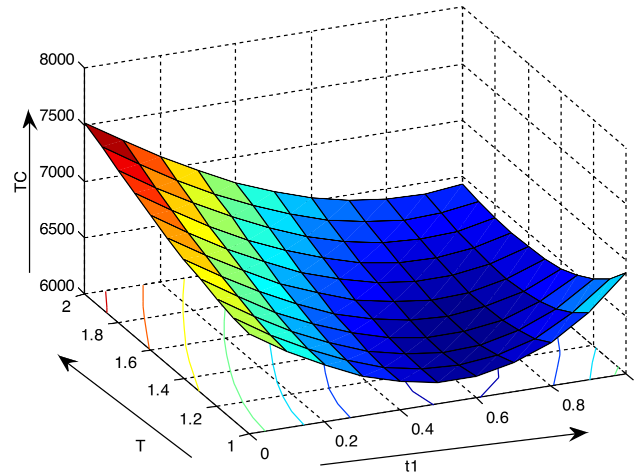

Fig. 7 Graphical observation of TC against t_1 and T

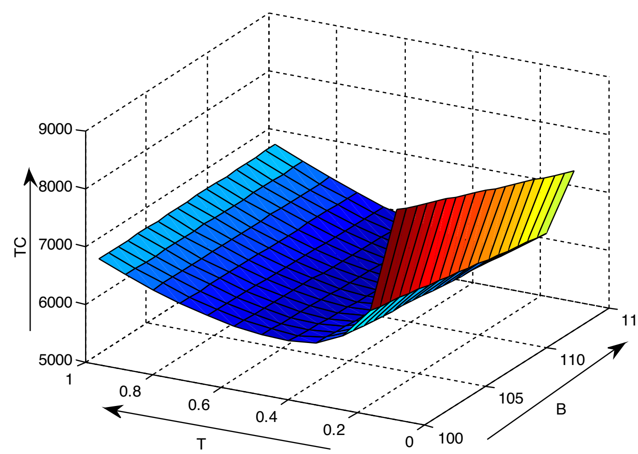

Fig. 8 Graphical observation of TC against B and T

Example 4. *Non-ending inventory procedure*

To investigate the zero ending situation as a special case of the developed non-ending inventory procedure, let us consider another hypothetical inventory procedure involving the values of several input parameters:

$A = \$600$/order, $a = 200$ units/month, $b = 0.5$, $c = 4$, $p = \$25$/unit, $c_p = \$15$/unit, $p_s = \$20$/unit, $h = \$\ 0.20$ /unit/-month,$\alpha = 25$ /unit/(month)^2, $t_d = 0.5$ month,$L = 0.6$ month, $\theta = 0.05$, $I_c = \$0.01$/month, $n = 15$, $\beta = 0.5$ and $U_w = 300$.

We use LINGO 14.0 for this example and find the following optimum solution:

$B^* = 0$ units, $S^* = 300$ units, $T^* = 1.5694$ months (47 days) and $TC^* = 5266.555$. Furthermore, the value of Ω as per our considered example is 27.9506. Since the value of Ω is positive, in accordance to the part (a) of the constructed criterion, the optimal inventory procedure terminates with zero ending.

6. Sensitivity analysis and managerial insights

This section inspects the influences of different input factors on the best solutions for both inventory procedures by accomplishing a tabular form sensitivity analysis in accordance to the aforementioned Example 2 and Example 3. These analyses have been implemented by altering the each input factor in relative steps of 10% (- 20%, – 10%, + 10%, +20%) and keeping static the all the other input factors at a time. Table 3 and Table 4 display the numerical outcomes of the analyses for the Example 2 and Example 3 respectively.

In accordance to the numerical outcomes in Table 3, the subsequent insights are drawn.

The practitioner's cost (TC^*) increases with an increment in the value of the parameters A,a, c, c_p, c_s, c_l, h, α, L, θ, β and decreases with an increment in the value of the parameters n,δ,b,p. Among the input parameters to accomplish prepayment, an increasing frequency of identical installment (n) helps the vendor to reduce the cost. On the other hand, decreasing input values of the allowed time length (L) and the percentage of purchase price (β) to complete the prepayment help the vendor to diminish the cost. Consequently, the practitioner should choose a smaller time duration, a smaller percentage of the purchase price and a higher identical installment frequency to complete the prepayment by negotiating with the supplier.

The inventory rotational time duration (T^*) reveals an upward inclination against the increment in the values of the parameters A,b,p,δ and a downward inclination against the increment in the value of the parameters a,c,c_p,c_s,c_l,h,α,θ. Among the input factors, the cost to place the order (A) has the sharpest upward inclination on the inventory rotational time length. The reason behind this insight is that when the cost to place an order is higher, the vendor tries to rise the number of purchase items and hence, the cycle time increases and the cost for placing order decreases in a unit time. On another hand, vending price of a unit item (p) has the downward consequences on the best inventory rotational time length as the market demand shows a descending inclination for an ascending vending price and hence, purchase products are sold for a longer period comparatively for a higher vending price. Furthermore, any significant consequences on T^* is not witnessed for varying (increasing or decreasing) the values of the factors L,n and β.

An acclivity on the best time duration of physical stock appearance in the storage (t_1^*) is appeared against the increasing values of the input factors A,b,p,c_s,c_l, while a declivity on t_1^* is witnessed against the rising values of a,c,c_p,h,α,θ,δ. The optimal t_1^*, similar to the best inventory rotation time length, is affected significantly by the cost for placing any order. However, the time sensitive cost to hold a unit item in the vendor's warehouse has considerably negative manner consequence on the optimum t_1^*. The reason is that if the products' time sensitive cost rises, the vendor's carrying cost also rises dramatically for a higher purchase quantity to store for a longer period in the warehouse. In addition, there is no change in the value of t_1^* when there is an increase or a decrease in the values of the parameters L,n and β.On account of the numerical outcomes in Table 4, the subsequent insights are studied.

When the cost to place the order (A), cost to purchase a unit item (c_p), constant term of demand (a), time associate coefficient of demand (c), both holding cost parameters (h and α), time associate coefficient of the decay rate (θ), allowed time length to complete prepayment (L) and the percentage of purchase price to accomplish prepayment (β) increase, then an acclivity is observed on the vendor's minimum cost (TC^*), while per unit vending price (p), coefficient of price in the demand (b) and frequency of identical installment to prepay (n) increase, then a declivity is observed on the vendor's minimum cost (TC^*). The per unit purchase price (c_p) badly affects the practitioner's cost (TC^*). As a result, the manager ought to discuss to reduce the value of c_p with the supplier to condense TC^* largely. In addition, when per item time sensitive carrying cost (α) rises, TC^* also rises significantly. This suggests to the practitioner to take some efficacious policies to diminish the time sensitive per unit carrying cost.

The optimum inventory rotational time length (T^*) is strictly rising for increasing the values of demand factor b and vending price (p). The reason of this observation is that for larger values of b and p, the products' demand falls and therefore, products will be sold for a longer period comparatively. Again, when the time associate coefficient (c) of the demand and time associate coefficient (θ) of the decay percentage increase, the optimum T^* falls gradually. The cause of this fact is that for larger value of c, the products' demand rises whereas for larger value of θ, the number of depleted items due to the deterioration increases to a great extent and as a result, the optimum T^* falls gradually. The best inventory rotational time interval becomes stable after a point of unit time's purchase price and before this stabilization, T^* increases up to that point of c_p. However, T^* remains stable up to a point of the constant parameter of holding price (h) and then, it falls slowly for increasing value of h. Consequently, c_p and h have impacts on T^* in an opposite manner. Furthermore, the value of T^* is immutable for any modification (increase or decrease) in the value of the parameters A,a,L,n and β.

Purchase price of a unit product (c_p) and the carrying cost parameters (h and α) have significant effects on the best remaining number of products B^*. As for a higher value of c_p, the practitioner wants to purchase a lower amount products. As a result, when a unit product's purchase price increases, there a value of c_p after which the optimal inventory procedure does not remain non-ending environment but becomes a zero ending inventory procedure. Similar observations are appeared for the carrying cost parameters h and α on the optimum B^*. For any varying (increasing or decreasing) on the values of the cost parameter to place the order (A) and demand parameter (a), the value of Ω always remain negative. Consequently, based on the results of the Theorem, there is some remaining items in the storage in the best inventory procedure for the practitioner. The best number of remaining products remains same for any changes of A, while B^* falls significantly when a increases. On another hand, for any fluctuating (increasing or decreasing) on the values of c,p,L,θ,η,β and b, the value of Ω is always positive and hence, the practitioner's best inventory procedure becomes zero ending environment.

Taking into account the aforementioned analysis and numerical results, the subsequent managerial insights are provided to the inventory managers in order to better improve of their business.

- When the per unit purchase price declines, the inventory manager can store products in the warehouse as much as possible and then, sell some products with a salvage value at the end of an inventory cycle.

Table 3 Sensitivity analysis for the input parameters of the inventory procedure when shortages perform.

Parameter	Original Value	New Values	t_1^*	T^*	TC^*
A	1000	800	0.6530	1.1414	5926.9410
		900	0.6768	1.2007	6012.3280
		1100	0.7191	1.3096	6171.6470
		1200	0.7382	1.3600	6246.5640
a	250	200	0.7460	1.3809	4971.2280
		225	0.7207	1.3138	5534.4130
		275	0.6793	1.2071	6649.7170
		300	0.6620	1.1636	7202.8860
b	0.5	0.40	0.6963	1.2503	6160.5960
		0.45	0.6975	1.2534	6127.1600
		0.55	0.7000	1.2598	6060.2520
		0.60	0.7012	1.2630	6026.7800
c	4	3.2	0.7022	1.2656	6082.3040
		3.6	0.7005	1.2611	6088.0200
		4.4	0.6970	1.2522	6099.3810
		4.8	0.6953	1.2478	6105.0280
p	30	24	0.6963	1.2503	6160.5960
		27	0.6975	1.2534	6127.1600
		33	0.7000	1.2598	6060.2520
		36	0.7012	1.2630	6026.7800
c_p	20	16	0.7045	1.2590	5143.9180
		18	0.7016	1.2578	5618.8290
		22	0.6959	1.2555	6568.5670
		24	0.6930	1.2544	7043.3940
c_s	10	8	0.6774	1.3403	6026.8470
		9	0.6889	1.2946	6062.6000
		11	0.7073	1.2246	6121.0660
		12	0.7147	1.1972	6145.3270
c_l	15	12	0.6976	1.2608	6090.1920
		13.5	0.6982	1.2587	6091.9590
		16.5	0.6993	1.2545	6095.4520
		18	0.7000	1.2518	6097.7510
h	0.25	0.20	0.7006	1.2572	6090.8340
		0.225	0.6997	1.2569	6092.2750
		0.275	0.6978	1.2563	6095.1460
		0.30	0.6969	1.2560	6096.5770
α	20	16	0.7607	1.2943	6043.2710
		18	0.7276	1.2740	6069.8710
		22	0.6734	1.2416	6115.2800
		24	0.6507	1.2284	6134.9450
L	0.5	0.4	0.6988	1.2566	6090.5880
		0.45	0.6988	1.2566	6092.1500
		0.55	0.6987	1.2566	6095.2740
		0.60	0.6987	1.2566	6096.8360
θ	0.03	0.024	0.7006	1.2578	6092.3740
		0.027	0.6997	1.2572	6093.0440
		0.033	0.6978	1.2560	6094.3780
		0.04	0.6957	1.2547	6095.9220
n	10	8	0.6987	1.2566	6094.0670
		9	0.6987	1.2566	6093.8700
		11	0.6988	1.2566	6093.5830
		12	0.6988	1.2566	6093.4750
β	0.6	0.48	0.6988	1.2566	6090.5880
		0.54	0.6988	1.2566	6092.1500
		0.66	0.6987	1.2566	6095.2740
		0.72	0.6987	1.2566	6096.8360
δ	0.04	0.032	0.6994	1.2535	6095.7160
		0.036	0.6991	1.2550	6094.7160
		0.044	0.6984	1.2582	6092.7040
		0.05	0.6979	1.2606	6091.1850

Table 4 Sensitivity analysis for the input parameters of the non-ending inventory procedure.

Parameter	Original Value	New Values	B^*	T^*	TC^*	Ω
A	1000	800	110.7200	0.80	6219.181	−0.3670000
		900	110.7200	0.80	6344.181	−0.3670000
		1100	110.7200	0.80	6594.181	−0.3670000
		1200	110.7200	0.80	6719.181	−0.3670000
a	250	200	150.7200	0.80	5337.515	−0.3669996
		225	130.7200	0.80	5903.348	−0.3670002
		275	90.72001	0.80	7035.015	−0.3670000
		300	70.72000	0.80	7600.848	−0.3669994
b	0.5	0.4	0.00	1.244611	6923.203	9.263429
		0.45	0.00	1.252253	6896.651	9.466573
		0.55	0.00	1.267816	6843.863	9.884238
		0.60	0.00	1.275740	6817.632	10.09896
c	4	3.2	0.00	1.262618	6859.933	9.744134
		3.6	0.00	1.261300	6865.074	9.708713
		4.4	0.00	1.258680	6875.323	9.638421
		4.8	0.00	1.257379	6880.432	9.603548
p	30	24	0.00	1.244611	6923.203	9.263429
		27	0.00	1.252253	6896.651	9.466573
		33	0.00	1.267816	6843.863	9.884238
		36	0.00	1.275740	6817.632	10.09896
c_p	20	16	110.7200	0.80	4966.706	−4.373600
		18	110.7200	1.2578	5717.944	−2.370300
		22	0.00	1.259987	7347.183	11.70545
		24	0.00	1.259987	7824.167	13.73743
h	0.25	0.20	0.00	1.259987	6862.666	9.609727
		0.225	0.00	1.259987	6866.435	9.641601
		0.275	110.7199	0.8000003	6474.320	−0.3469917
		0.30	110.7200	0.8000002	6479.460	−0.3269922
α	20	16	110.7200	0.8000001	6190.738	−1.646999
		18	0.00	1.259987	6743.298	8.069979
		22	0.00	1.259987	6997.108	11.27697
		24	0.00	1.259987	7124.013	12.88047
L	0.5	0.4	0.00	1.259987	6868.632	9.666781
		0.45	0.00	1.259987	6869.418	9.670128
		0.55	0.00	1.259987	6870.989	9.676822
		0.60	0.00	1.259987	6871.775	9.680170
θ	0.03	0.024	0.00	1.260589	6867.673	9.599008
		0.027	0.00	1.260288	6868.939	9.636239
		0.033	0.00	1.259687	6871.466	9.710697
		0.04	0.00	1.259388	6872.727	9.747914
n	10	8	0.00	1.259987	6870.382	9.674236
		9	0.00	1.259987	6870.283	9.673813
		11	0.00	1.259987	6870.139	9.673199
		12	0.00	1.259987	6870.084	9.672968
β	0.6	0.48	0.00	1.259987	6868.632	9.666781
		0.54	0.00	1.259987	6869.418	9.670128
		0.66	0.00	1.259987	6870.989	9.676822
		0.72	0.00	1.259987	6871.775	9.680170

- If the holding cost per unit time for a unit product increases, the decision maker can shrink the optimal inventory cycle and sell the remaining unsold products with a salvage value at the end. This strategy helps the manager to reduce the total carrying cost significantly.
- When the time varying cost coefficient in the per unit holding cost decreases, the industry manager can store the highest possible number of products in the warehouse. In this case, he/she should not sell all the products at the regular selling price rather than sell some products with a salvage value at the end to optimize the total cost.
- As there is a cost (loan cost) on the advance payment portion, the decision maker can benefit economically by fulfilling the prepayment for the possible minimum portion of the total purchase price after negotiating with the supplier.
- In order to reduce the total cost under a hybrid cash-advance payment, the manager should accomplish the advance payment portion through the possible maximum number of installments after negotiating with the supplier.
- When the ordering cost for each inventory cycle decreases, the total cost of the industry manager shrinks largely. Consequently, the manager is suggested to reduce the ordering

cost following several effective policies such as introducing technical advancements or financing to diminish the ordering cost.

7. Conclusion

Companies are seeking a proper inventory strategy for perishable items not only to increasing profit but also to ensure the proper use of natural resources and protection of the environment. In this study, two sustainable inventory procedures for both delayed and time sensitive decay items are analyzed where a mixed cash-advance payment strategy is adopted. To make the problems more representative, not only time associate demand but also time associate per unit carrying cost and capacity limitation in the practitioner's warehouse are incorporated. In the first inventory procedure, a stock out situation is handled by adopting waiting time associated backlogging rate while a non-ending case with salvage values is explored in the other model. The utmost aim is to minimize the practitioner's cost for both inventory procedures. However, due to the exceeding nonlinearity, closed form solutions are not possible to achieve for both cases, and hence, solutions are achieved by utilizing some classic techniques. A key criterion is constructed to decide whether the best inventory scheme in a non-ending inventory problem involves the remaining items being in the stock or not. Utilizing the developed criterion, an algorithmic methodology is provided to determine the best inventory scheme for a non-terminating inventory problem, and this methodology is examined by solving several hypothetical inventory procedures. Furthermore, some insights are made based on numerical findings and sensitivity analyses. For instance, the practitioner ought to choose a smaller time duration, a smaller percentage of the purchase price, and a higher identical installment frequency to complete the prepayment by negotiating with the supplier. Secondly, the industry manager can store the highest possible number of products in the warehouse when the per unit purchase price declines and then, sell some products with a salvage value at the end of an inventory cycle. Thirdly, sale of the products at the regular price is not always economical for the decision maker. more concretely, when the time varying cost coefficient in the per unit holding cost decreases, then the industry manager can store the highest possible number of products in the warehouse and sell some products with a salvage value at the end of the cycle.

The demand pattern in the current study is limited to consider either increasing or decreasing linearly with respect to storage time. However, many products, like processed foods, vegetables, fruit, etc., have greater demand at the initial period than at the end of an inventory cycle. On the other hand, some products, such as sugar, rice, oil, water, salt, and so on, may have lower demand at the start of an inventory cycle than at the end. Both situations cannot be studied with the linear form of time sensitive demand. Consequently, adopting non-linear time sensitive demand instead of linear time sensitive demand is more appropriate in reality.

Finally, anyone can study the work further by determining the best frequency of identical installment to minimize the cost after considering some costs for each installment. Secondly, the best pricing strategy can also be explored, taking into account profit optimizing instead of cost optimizing inventory procedures (Khan et al. [10]). Also, trade credit opportunities (Ahmed et al. [33]) (single level, two level or partial), another realistic feature, can be introduced in this work. Anyone may further explore this work by adopting all the inventory costs in interval valued (Rahman et al. [11]) or fuzzy valued (Omair et al. [22]).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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