$M^X/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1/G(BS)/V_s$ QUEUE UNDER CONTROLLED ADMISSIBILITY POLICY

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Abstract: *In this paper, we propose to study such a model which deals with the aspects concerning the control of the arrival process. The paper deals with $M^X/G/1$ queueing system with two types of*

$$E[L_b]=\frac{rE[X]\{\theta_1 E[B_1]+\theta_2 E[B_2]\}\{1+\lambda\alpha E[V]\}}{1-r\lambda E[X]\{\theta_1 E[B_1]+\theta_2 E[B_2]\}}$$

heterogeneous services and Bernoulli vacation schedule under a controlled admissibility policy of arriving batches. There is a single server who provides two types of parallel general heterogeneous services (one of which has to be chosen by each customer) to the customer on FCFS basis. Before starting the service, each customer has option to choose first service with probability θ_1 or the second service with probability θ_2. The server's vacations are based on Bernoulli schedule under a single vacation policy where after completion of service (of any phase), the server either goes for a vacation of random length with probability α or may continue to serve the next customer with probability $(1-\alpha)$, if any. Under a controlled admissibility policy it is assumed that not all batches are allowed to join the system at all times. We obtain explicit queue size distribution at random epoch as well as at the departure epoch under the steady state conditions. In addition, some performance measures such as expected queue size and expected waiting time of a customer are also obtained. The numerical results for various performance measures are summarized displayed via graphs.

Keywords: $M^X/\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}/1/G(BS)/V_s$ *queue, Heterogeneous services, Controlled admissibility policy, Bernoulli schedule, Vacation, Phase service, Supplementary variable, Queue Size.*

Date of submission : 06.03.2013 Date of acceptance : 27.07.2013

1. Introduction

A wide class of policies for governing the vacation mechanism has been discussed in literature. The classical vacation scheme with Bernoulli vacation discipline was introduced and studied by Keilson and Servi [1]. Ke, J.C., Wu, C.H. and Pearn, W.L. [2] estimated an infinite-capacity M/M/c queueing system with modified Bernoulli vacation under a single vacation policy.

This model also belongs to a class of systems where the service discipline involves more than one service and which has been receiving a lot of attention recently. Park, H.M., Kim, J.S. and Chae, K.C. [3] analyzed two phase queueing system with fixed batch size policy. The optimal control policy of a batch arrival queue with two phases of service and Bernoulli vacation schedule is considered by Choudhury and Tadj [4].

Some aspects of batch arrival Bernoulli vacation models with restricted admissibility in to the system were studied by Madan and Abu-Dayyeh [5]. Recently Choudhury [6] examined a $M^X/G/1$ queue with setup time under a restricted admissibility policy with Bernoulli vacation schedule.

In the present paper, we consider a single server queueing system in which arrival occurs according to compound Poisson Process with batches of random size X. Single server provides two kinds of parallel general heterogeneous service to the units one by one on FCFS basis. Further it is assumed that not all batches are allowed to join the system at all time. The concept of Bernoulli vacation schedule is also incorporated. The paper is organized as follows. In section 2, we define the underlying assumptions and notations of the system under study and also construct the steady state equations. The analysis based on supplementary variables and generating function approach, is given in section 3. The queue size distributions at random epoch and departure epoch are obtained in sections 4. Mean queue size at random epoch and departure epoch are determined in sections 5. Mean busy period is calculated in section 6. Section 7 is meant for sensitivity analysis .In the last section 8, the conclusions are drawn.

2. Description of the Model

We consider an $M^X/G/1$ queueing system with the following assumptions:

- The customers arrive at the system according to a compound Poisson process with random batch size denoted by variable 'X'.
- Arriving batch join the queue with probability $r(0 \le r \le 1)$ when server is in busy state and arriving batch join the queue with probability $p(0 \le p \le 1)$ when server is in vacation state. Let λ be the mean arrival rate of the customers.
- There is a single server who provides two kinds of general heterogeneous services to the customers on a first come first served (FCFS) basis. Just before a service starts a customer has option to choose one of the two types of services.
- Customer may choose first service with probability θ_1 or the second service with probability θ_2.
- As soon as the service of a customer is completed, the server may take a vacation of random duration with probability α or else with probability $(1-\alpha)$, he may continue servicing the next customer, if any. On completion of vacation period, the server must be back to the system even if there is no customer to serve.
- We assume that the service time random variable $S_j(x)$ of the j^{th} type of service follows a general probability law with $B_j(s_j)$ as the distribution function, $b_j(s_j)$ as probability density function and Laplace Stieltjes transform *(LST)* of $S_j(x)$ is $B_j^*(s)$ with finite k^{th} moment $E(S_j^k)$ $(k=\ge 1)$ of the service time, $j = 1, 2$.
- Next we assume that vacation random variable V follows a general probability distribution with distribution function $V(x)$ of the server follows a general probability law with distribution function $V(x)$, Laplace Stieltjes transform *(LST)* is $V^*(s)$ with finite moment $E(V^k)$, $(k=\ge 1)$.

Let $N_Q(t)$ be the queue size at time 't'. To make it Markov process we introduce supplementary variables $B_j^0(t)$ and $V^0(t)$ where $B_j^0(t)$ be the elapsed service time at time 't' and $V^0(t)$ be the elapsed vacation time at time 't'. Let the status of the server at time t

$$Y(t) = \begin{cases} 0, & \text{server is idle at time } t \\ 1, & \text{server is busy with first type of service at time } t \\ 2, & \text{server is busy with second type of service at time } t \\ 3, & \text{server is on vacation at time } t \end{cases}$$

is denoted by $Y(t)$ as

Define the limiting probabilities as:

$$I_0 = Lim_{t\to\infty} \Pr[N_Q(t)=0,\ Y(t)=0]$$

$$P_{1,n}(x)dx = Lim_{t\to\infty} \Pr[N_Q(t)=n,\ Y(t)=1, x < B_1^0(t) \le x+dx]$$

$$P_{2,n}(x)dx = Lim_{t\to\infty} \Pr[N_Q(t)=n, Y(t)=2, x < B_2^0(t) \le x+dx]$$

$$Q_n(x)dx = Lim_{t\to\infty} \Pr[N_Q(t)=n, Y(t)=3, x < V^0(t) \le x+dx]$$

Hazard rates are given by;

$$\mu_j(x)dx = \frac{dB_j(x)}{[1-B_j(x)]},\quad j=1,2\ ;\ \eta(x)dx = \frac{dV(x)}{[1-V(x)]}$$

Steady State Equations

Now Chapman Kolmogorov equations governing the models are constructed as follows:

$$\frac{d}{dx}P_{n,1}(x)+(\lambda+\mu_1(x))P_{n,1}(x)=\lambda(1-r)P_{n,1}(x)$$

$$+r\lambda\sum_{i=1}^{n}a_iP_{n-i,1}(x);\ x>0,\ n\geq 1 \qquad (1)$$

$$\frac{d}{dx}P_{n,2}(x)+(\lambda+\mu_2(x))P_{n,2}(x)=\lambda(1-r)P_{n,2}(x)$$

$$+r\lambda\sum_{i=1}^{n}a_iP_{n-i,2}(x);\ x>0,\ n\geq 1 \qquad (2)$$

$$\frac{d}{dx}Q_n(x)+(\lambda+\eta(x))Q_n(x)=\lambda(1-p)Q_n(x)$$

$$+p\lambda\sum_{i=1}^{n}a_iQ_{n-i}(x);x>0,\ n\geq 1 \qquad (3)$$

$$\frac{d}{dx}Q_0(x)+(\lambda+\eta(x))Q_0(x)=\lambda(1-p)Q_0(x)$$

$$;\ x>0 \qquad (4)$$

$$\lambda I_0=\lambda(1-r)I_0+\int_0^\infty Q_0(x)\eta(x)\,dx$$

$$+(1-\alpha)\left\{\int_0^\infty P_{n+1,1}(x)\,\mu_1\,dx+\int_0^\infty P_{n+1,2}(x)\,\mu_2(x)\,dx\right\} \qquad (5)$$

These equations are to be solved subject to the following boundary conditions:

$$P_{n,1}(0)=(1-\alpha)\,\theta_1\left\{\int_0^\infty P_{n+1,1}(x)\,\mu_1(x)\,dx\right.$$

$$\left.+\int_0^\infty P_{n+1,2}(x)\,\mu_2(x)\,dx\right\}+\theta_1\int_0^\infty \eta(x)\,Q_n(x)\,dx$$

$$+r\lambda\theta_1a_nI_0;\ n\geq 1 \qquad (6)$$

$$P_{n,2}(0)=(1-\alpha)\,\theta_2\left\{\int_0^\infty P_{n+1,1}(x)\,\mu_1(x)\,dx+\int_0^\infty P_{n+1,2}(x)\,\mu_2(x)\,dx\right\}$$

$$+\theta_2\int_0^\infty \eta(x)\,Q_n(x)\,dx+r\lambda\theta_2a_nI_0;\ n\geq 1 \qquad (7)$$

$$Q_n(0)=p\alpha\left\{\int_0^\infty P_{n+1,1}(x)\,\mu_1(x)\,dx+\int_0^\infty P_{n+1,2}(x)\,\mu_2(x)\,dx\right\} \qquad (8)$$

The normalizing condition yields:

$$I_0+\sum_{j=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty P_{n,j}(x)dx+\sum_{n=0}^{\infty}\int_0^\infty Q_n(x)\,dx=1 \qquad (9)$$

Define the following generating functions:

$$P_j(x;z)=\sum_{n=1}^{\infty}z^nP_{n,j}(x),\ j=1,2\ ;$$

$$P_j(0;z)=\sum_{n=1}^{\infty}z^nP_{n,j}(0),\ j=1,2\ ;$$

$$Q(x;z)=\sum_{n=0}^{\infty}z^nQ_n(x)\ ;\ Q(0;z)=\sum_{n=0}^{\infty}z^nQ_n(0)$$

3. The Analysis

In this section, we obtain joint and marginal generating functions of queue size as follows:

3.1 Joint Probability Generating Functions:

The joint probability generating functions when the server is busy in providing first type of service, when the server is busy in providing second type of service and on vacations respectively, are given by:

$$P_1(x,z)=P_1(0,z)\exp[-r\lambda(1-X(z))x]\{1-B_1(x)\} \qquad (10)$$

$$P_2(x,z)=P_2(0,z)\exp[-r\lambda(1-X(z))x]$$
$$\{1-B_2(x)\} \qquad (11)$$

$$Q(x,z)=Q(0,z)\exp[\{-p\lambda(1-X(z)\}x]$$
$$\{1-V(x)\} \qquad (12)$$

3.2 Marginal Generating Functions

Multiply (6)-(8) by suitable powers of z, take summation over all possible values of n and simplify, we get:

$$\left\{z-(1-\alpha)\theta_1 B_1^*(r\lambda(1-X(z)))\right\}P_{1,0}(z)$$
$$=(1-\alpha)\theta_1 B_2^*(r\lambda(1-X(z)))\,P_{2,0}(z)$$
$$+z\theta_1 V^*(p\lambda(1-X(z)))\,Q(0,z)$$
$$+\ zr\lambda\theta_1(X(z)-1)I_0 \qquad (13)$$

$$\left\{z-(1-\alpha)\theta_2 B_2^*(r\lambda(1-X(z)))\right\}P_{2,0}(z)$$
$$=(1-\alpha)\theta_2 B_1^*(r\lambda(1-X(z)))\,P_{1,0}(z)$$
$$+z\theta_2 V^*(p\lambda(1-X(z)))\,Q(0,z)$$
$$+\ zr\lambda\theta_2(X(z)-1)I_0 \qquad (14)$$

$$zQ(0,z)=\alpha\left\{P_1(0,z)\right\}\left\{B_1^*(r\lambda(1-X(z)))\right.$$
$$\left.+P_2(0,z)\,B_2^*(r\lambda(1-X(z))\right\} \qquad (15)$$

where

$$B_1^*(r\lambda(1-X(z))=\int_0^\infty e^{-r\lambda(1-X(z))x}\,dB_1(x)$$

$$B_2^*(r\lambda(1-X(z))=\int_0^\infty e^{-r\lambda(1-X(z))x}\,dB_2(x)$$

$$V^*(p\lambda(1-X(z))=\int_0^\infty e^{-p\lambda(1-X(z))x}\,dV(x)$$

Solve (13) & (14) for $P_1(0,z)$ and $P_2(0,z)$, we have

$$P_1(0,z)=\frac{r\lambda z\theta_1(1-X(z))I_0}{\zeta(z)} \qquad (16)$$

$$P_2(0,z)=\frac{r\lambda z\theta_2(1-X(z))I_0}{\zeta(z)} \qquad (17)$$

Using (16) and (17) in (15), we get

$$Q(0,z)=\frac{r\lambda\alpha\left\{\theta_1 B_1^*\{r\lambda(1-X(z))\}+\theta_2 B_2^*\{r\lambda(1-X(z))\}\right\}}{\zeta(z)}$$
$$\times(1-X(z))I \qquad (18)$$

Where

$$\zeta(z)=\left\{(1-\alpha)+\alpha V^*\{p\lambda(1-X(z))\}\right\}$$
$$\left\{\theta_1 B_1^*\{r\lambda(1-X(z))\}+\theta_2 B_2^*\{r\lambda(1-X(z))\}\right\}-z$$

By using Equation (16)-(18) we obtain the marginal generating functions when the server is busy in providing first type of service, when the server is busy in providing second type of service and on vacations respectively, as follow:

$$P_1(z)=\int_0^\infty P_1(x,z)dx=\frac{z\theta_1\left\{1-B_1^*(r\lambda(1-X(z)))\right\}I_0}{\zeta(z)} \qquad (19)$$

$$P_2(z)=\int_0^\infty P_2(x,z)dx=\frac{z\theta_2\left\{1-B_2^*(r\lambda(1-X(z)))\right\}I_0}{\zeta(z)} \qquad (20)$$

$$Q(z)=\int_0^\infty Q(x,z)dx$$
$$=\frac{\alpha\left\{\theta_1 B_1^*\{r\lambda(1-X(z))\}+\theta_2 B_2^*\{r\lambda(1-X(z))\}\right\}}{\zeta(z)}$$
$$\times\left\{1-V^*(p\lambda(1-X(z)))\right\}I_0 \qquad (21)$$

Where

$$\zeta(z)=\left\{(1-\alpha)+\alpha V^*\{p\lambda(1-X(z))\}\right\}$$
$$\left\{\theta_1 B_1^*\{r\lambda(1-X(z))\}+\theta_2 B_2^*\{r\lambda(1-X(z))\}\right\}-z$$

3.3 Steady State Probabilities:

To determine I_0, we use normalizing condition equivalent to $I_0+P_1(1)+P_2(2)+Q(1)=1$ and get the steady state probability that the server is in idle state, as

$$I_0=1-\phi \qquad (22)$$

where $\phi=\lambda E[X]\{rE[B_1]+rE[B_2]+p\alpha E[V]\}<1$ is the utilization factor of this system and $E[X]$ is mean size of an arriving batch.

Consequently,

Pr[The server is busy with first kind service]=

$$Lim_{z\to 1}P_1(z)=r\lambda\,E(x)\left[\theta_1\,E[B_1]+\theta_2\,E[B_2]\right] \qquad (23)$$

Pr[The server is busy with second kind service]=

$$Lim_{z\to 1} P_1(z) = r\lambda\, E(x)\left[\theta_1\, E[B_1] + \theta_2\, E[B_2]\right] \quad (24)$$

Pr [The server is on vacation]=

$$Lim_{z\to 1} Q(z) = \alpha\, p\lambda\, E(x)\, E[V] \quad (25)$$

4. Queue Size Distribution

Let $P_Q(z)$ denote the steady state PGF of the queue size distribution at random epoch of $M^X/G/1$ queueing system with two types of heterogeneous services and Bernoulli vacation schedule under a controlled admissibility policy of arriving batches , where

$$P_Q(z) = I_0 + P_1(z) + P_2(z) + zQ(z) \quad (26)$$

Therefore

$$P_Q(z) = \frac{(1-z)(1-\phi)\left\{(1-\alpha)+\alpha V^*(p\lambda(1-X(z)))\right\}}{\left\{(1-\alpha)+\alpha V^*(p\lambda(1-X(z)))\right\}-z} \times \frac{\left\{\theta_1\, B_1^*\{r\lambda(1-X(z))\}+\theta_2\, B_2^*\{r\lambda(1-X(z))\}\right\}}{\left\{\theta_1\, B_1^*\{r\lambda(1-X(z))\}+\theta_2\, B_2^*\{r\lambda(1-X(z))\}\right\}} \quad (27)$$

4.1 Queue size distribution at departure epoch

Following an argument of PASTA, we state that a departing customer will see 'k' customers in the queue if and only if there were $(k+1)$ customers in the queue just before his departure. Let $a_k;\ k \ge 0$ is the probability that there are k customers in the queue at departure epoch. Then we have

$$a_k = b_0(1-\theta)\left[\int_0^\infty \mu_1(x)\,P_{k+1,1}(x)\,dx + \int_0^\infty \mu_2(x)\,P_{k+1,2}(x)\,dx\right] + b_0\int_0^\infty \eta(x)\,Q_k(x)\,dx;\ k\ge 0 \quad (28)$$

$$a(z) = \sum_{k=0}^{\infty} z^k a_k \quad (29)$$

where b_0 is a normalizing constant. Let $a(z)$ be the PGF of $a_k;\, k \ge 0$, then using equation (16)-(18), we have

$$a(z) = \frac{b_0 r\lambda I_0(1-X(z))\left\{(1-\alpha)+\alpha V^*(p\lambda(1-X(z)))\right\}}{\left\{(1-\alpha)+\alpha V^*(p\lambda(1-X(z)))\right\}} \times \frac{\left\{\theta_1\, B_1^*\{r\lambda(1-X(z))\}+\theta_2\, B_2^*\{r\lambda(1-X(z))\}\right\}}{\left\{\theta_1\, B_1^*\{r\lambda(1-X(z))\}+\theta_2\, B_2^*\{r\lambda(1-X(z))\}\right\}-z} \quad (30)$$

By using normalizing condition $a(1)=1$, we get

$$b_0 = \frac{1-\phi}{r\lambda\, I_0\, E[X]} \quad (31)$$

therefore

$$a(z) = \frac{(1-\phi)(1-X(z))\left\{(1-\alpha)+\alpha V^*(p\lambda(1-X(z)))\right\}}{E[X]\left\{(1-\alpha)+\alpha V^*(p\lambda(1-X(z)))\right\}} \times \frac{\left\{\theta_1\, B_1^*\{r\lambda(1-X(z))\}+\theta_2\, B_2^*\{r\lambda(1-X(z))\}\right\}}{\left\{\theta_1\, B_1^*\{r\lambda(1-X(z))\}+\theta_2\, B_2^*\{r\lambda(1-X(z))\}\right\}-z} \quad (32)$$

5. Operating Characteristics

In this section, we derive the average queue length and average waiting time of a customer in the queue.

Let L_Q be the mean queue size at random epoch, then

$$L_Q = \left(\frac{dP_Q(z)}{dz}\right)_{z=1}$$

$$= \phi + \lambda^2 (E[X])^2$$

$$= \phi + \frac{\lambda^2 E^2[X][r^2(\theta_1 E[B_1^2] + \theta_2 E[B_2^2])]}{2(1-\phi)}$$

$$\frac{+2\theta\, prE[V]((\theta_1 E[B_1] + \theta_2 E[B_2])) + \theta\, p^2 E[V^2]}{2(1-\phi)}$$

$$+\frac{E[X(X-1)]}{2E[X]} \quad (33)$$

Let L_D be the mean queue size at departure epoch, then

$$L_Q = \left(\frac{da(z)}{dz}\right)_{z=1}$$

$$= L_Q + \frac{E[X(X-1)]}{2E[X]} \quad (34)$$

6. Mean Busy Period

In this section we obtain the mean busy period for our model

$M^X/(G_1, G_2)/1/G(BS)/V_s$

queue under controlled admissibility policy. We define the busy period as the length of time during which the server remains busy and this continues till the instant when the server becomes free again. This busy period is equivalent to ordinary busy period, vacation period plus an idle period.

Thus we define

(a) L_I = length of generalized idle period (vacation plus idle period)

(b) L_b = Length of busy period

L_I and L_b generate an alternating renewal process, therefore we may write

$$\frac{E(L_b)}{E(L_I)} = \frac{\Pr[L_b]}{1-\Pr[L_b]}$$

$$\Pr[L_b] = P_1(1) + P_2(1) = r\lambda E[X]\{\theta_1 E[B_1] + \theta_2 E[B_2]\}$$

and $E[L_I] = \frac{1}{\lambda} + \alpha E[V] = \frac{1+\lambda\alpha E[V]}{\lambda}$

$$E[L_b] = \frac{rE[X]\{\theta_1 E[B_1] + \theta_2 E[B_2]\}\{1+\lambda\alpha E[V]\}}{1 - r\lambda E[X]\{\theta_1 E[B_1] + \theta_2 E[B_2]\}}$$

Fraction of time when the server remains in idle and on vacation (i.e. generalized idle state) is given by T_0, where

$$T_0 = \frac{E[L_I]}{E[L_I]+E[L_b]}$$

$$= \frac{1+\lambda\alpha E[V]}{1+\lambda\alpha E[V] + r\lambda^2 E[X]\{\theta_1 E[B_1] + \theta_2 E[B_2]\}}$$

7. Sensitivity Analysis

In this section, we validate our analytical results by taking numerical examples. The sensitivity analysis is performed to visualize the effect of different parameters on the average queue length.

In figures 1-2, we observed that as λ increases, there is remarkable increase in the queue length; the impact is more prominent for higher values of λ in comparison to lower values of λ. From fig. 1 we notice that higher service rates have a significant impact on the queue length as it leads to considerably decrement in the queue length. For higher values of vacation rate, the lower values of average queue length can be seen in fig. 2.

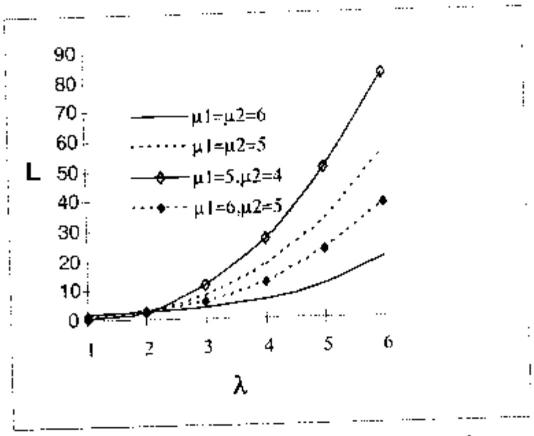

Figure 1: Queue length vs λ

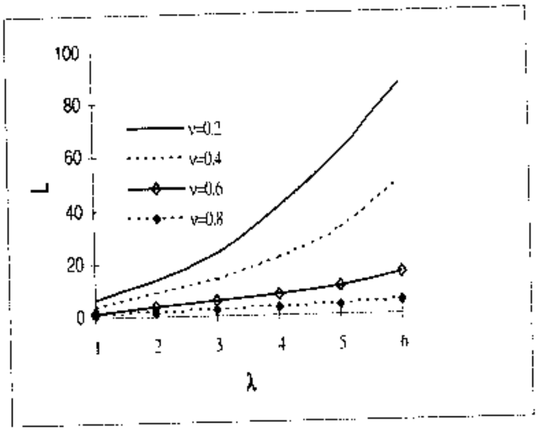

Figure. 2: Queue length vs λ

8. Conclusions

In this investigation, we have obtained explicit queue size distributions for $M^X/G/1$ queue with Bernoulli vacation schedule under restricted admissibility policy with two types of heterogeneous services. In many congestion situations just before a service starts, the customer has the option to choose one of two types of services. Further our model assumes that the server vacations are based on Bernoulli schedule which means that just after completing a service selected by the customer, the server may take vacation of random length or may continue staying in the system. The concepts of Bernoulli schedule vacation, batch arrival and restricted admissibility policy have been incorporated together in our queueing model which has potential applicability in manufacturing, computer and communication systems, etc..

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