ON DISTRIBUTIVE PSEUDOCOMPLEMENTED LATTICE

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Abstract: *In lattice theory there are different classes of lattices known as variety of lattices. Distributive pseudocomplemented lattice is one of the large variety. Throughout this paper we discuss pseudocomplemented lattice. Pseudocomplemented lattice were introduced by H. Lakser [2], [4], K.B.Lee [6]. In this paper we have studied pseudocomplemented lattices and generalized Stone lattice.*

Keywords: *pseudocomplemention, Dense lattice, generalized Stone lattice, Sectionally pseudocomplemented Lattices*

1. Preliminaries

1.1 Pseudocomplement: Let L be a bounded distributive lattice, let $a \in L$ an element $a^* \in L$ is called a pseudocomplement of a in L if the following conditions holds: (i) $a \wedge a^* = 0$ (ii) $\forall x \in L$, $a \wedge x = 0$ implies that $x \leq a^*$.

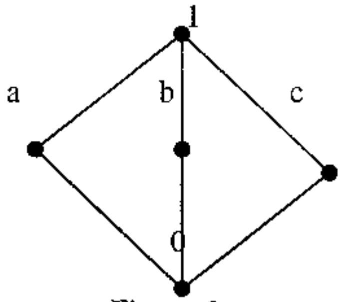

Figure-1

1.2 Pseudocomplemented Lattice: A bounded lattice L is called a pseudocomplemented lattice if its every element has a pseudocomplement.

Example 1.

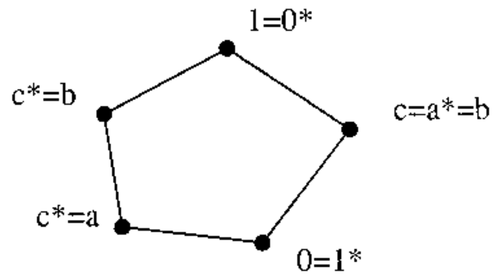

Figure-2

The lattice $L = \{0, a, b, c, 1\}$ shown in fig (2) is pseudocomplemented.

1.3 Lattice with pseudocomplementation: An algebra, $< L, \wedge, \vee, *, 0, 1 >$ where $\wedge, \vee$ are binary operation * is a unary operation and 0,1 are nullary operations is called lattice with pseudocomplemention if $< L, \wedge, \vee, *, 0, 1 >$ is bounded lattice, i.e. $\forall\ a \in L$ there exists a^* such that $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \wedge a^* = x, \forall\ x \in L$.

1.4 Pseudocomplemented Distributive Lattice:

A bounded distributive lattice L is called a pseudocomplemented distributive lattice if its every element has pseudo complemented.

Example 2.

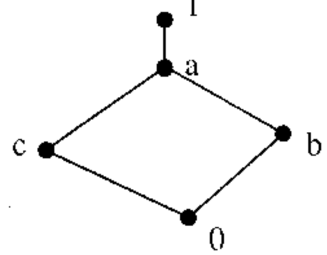

Figure-3

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Consider the finite distributive lattice of fig. (1). As a distributive lattice it has twenty five sublattices and eight congruences; as a lattice with pscomplementation has three sub-algebras and five congruences.

Boolean Lattice: A complemented distributive lattice is called Boolean lattice.

Stone Lattice: A distributive pseudocomplemented lattice L is called stone lattice if for $a \in L$, $a^* \vee a^{**} = 1$

Example 3. Every Boolean lattice is stone lattice but converse is not true.

Stone algebra: A complemented distributive lattice is called a Stone algebra if for each $a^* \vee a^{**} = 1$

Generalized Stone Lattice: A lattice L with 0 is called generalized Stone lattice if $(x]^* \vee (x]^{**} = L$ for each $x \in L$.

Sectionally pseudocomplemented Lattices: A lattice L with 0 is called sectionally pseudocomplemented if interval $[0, x]$ for each $x \in L$ is pseudocomplemented.

Note: Every finite distributive lattice is sectionally pseudocplemented.

Following example of a distributive lattice with 0 which is not sectionally pseudocomplemented.

In R^2 consider the set:

$E = \{(0, y)/0 \leq y < 5\} \cup \{(2, y)/$
$0 \leq y < 5\} \cup \{(3,5), (4,5), (3,6)\}$

Define the partial ordering $\leq$ on E by $(x, y) \leq (x_1, y_1)$ if and only if $x \leq x_1$ and $y \leq y_1$ here E is clearly a distributive lattice. This is not a lattice as the supremum of (3,6) and (4,5) does not exist. Consider the interval $[0, p]$ observe that in this interval (2,0) has no relative pseudocomplemented. So $(E, \leq)$ is not sectionally pseudocomplemented.

2. Normal Lattice: A distributive lattice L with 0 is called normal lattice if each prime ideal of L contains a unique minimal prime ideal. Equivalently, L is called normal if each prime filter of L is contained in a unique ultrafilter (maximal and proper of L).

Dense Lattice: A lattice L with 0 is called Dense lattice if $(x]^* = (0]$ for each $x \neq 0$ in L.

Lemma 1. Let L be a distributive lattice with 0. Let $0 \leq x \in L$ and the interval $[0, x]$, is complemented. If y^* is the relative complemented of y in $[0, x]$, then $(y^*] = (y]^* \wedge (x](x]$

and $(y^{**}] = (y] \wedge (x]$. □

Lemma 2. Let L be a distributive and I be any ideal $((r] \wedge I]^* \wedge (I] = I^* \wedge (r]$. □

Theorem 1. A distributive lattice L with 0 is a generalized Stone lattice if and only if each interval $[0, x]$, $0 < x \in L$, L is Stone lattice with 0. For any $r \in L$

Proof: Let L with 0 be a generalized stone and let $P \in [0, x]$.

Then $(P]^* \vee (P]^{**} = L$. So $x \in (P]^* \vee (P]^{**}$ implies $x = r \vee I$,

for some $r \in (P]^*, I \in (P]^{**}$

Now $r \in (P]^*$ implies $r \wedge P = 0$, also $0 \leq r \leq x$. Suppose $t \in [0, x]$ such that $t \wedge P = 0$, then $t \in (P]^*$ implies $t \wedge I = 0$ Therefore,
$t \wedge x = t \wedge (r \vee I) = (t \wedge r) \vee (t \wedge I)$
$= (t \wedge r) \vee 0 = t \vee r$

implies $t = t \wedge r$ implies $t \leq r$

So, r is the relative complement of P in $[0, x]$, *i.e.*, $r = P$.

Since $I \in (P]^{**}$. and $r \in (P]^*$, so $I \wedge r = 0$. Let $q \in [0, x]$ such that $q \wedge r = 0$ Then as $x = r \vee I$ so $q \wedge x = (q \wedge r) \vee (q \wedge I)$

Implies $q = q \leq I$ implies $q \leq I$

Hence. I is the relative complement of $r = P*$ in $[0,x], i.e., I = P**$ implies $x = r \vee I = P* \vee P**$. Thus $[0,x]$, is a stone lattice.

Conversely, suppose $[0,x]$, $0 < x \in L$ is a stone lattice. Let $P \in L$,

Then $P \wedge x \in [O,P]$ Since $[O,P]$ is a stone lattice, then

$(P \wedge x)* \vee (p \wedge x)** = P$

where $(P \wedge x)*$ is the relative complement of $(P \wedge x)$ in $[O,P]$

Therefore

$P \in ((p] \cap (p \wedge x]) \vee ((p] \cap (P \wedge x]**)$

So, we can take $P = r \vee 1$,

for $r \in (P \wedge x]*$, $1 \in (p \wedge x]**$

Now, $r \in (P \wedge x]*$

implies $r \wedge P \wedge x = 0$

implies $r \wedge x = 0$ implies

$r \in (x]**$ and $1 \in (P \wedge x]**$

Now $P \wedge x \leq x$

implies $(P \wedge x]** \subseteq (x]**$

And so $1 \in (x]**$

Therefore $P = r \vee 1 \in (x]* \vee (x]**$

and so $L \subseteq (x]* \vee (x]**$

But $(x]* \vee (x]** \subseteq L$ is obvious.

Hence $(x]* \vee (x]** = L$

and S on L is generalized Stone. □

Following theorem is a generalization of [4, Proposition 5.5(b)]

Theorem 2 Suppose L be a distributive lattice with 0. If L is generalized Stone, then it is normal.

Proof: Let P and Q be two minimal prime ideals of L. Then P,Q are unordered.

Let $x \in P$, Then $(x] \wedge (x]* = (0] \subseteq Q$ implies $(x]* \subseteq Q$. Since P is minimal, so $(x]** \subseteq P$. Again, as L is generalized stone, so $(x]* \vee (x]** = L$.

This implies $P \vee Q = L$ and so L is normal. □

Theorem 3. A sectionally Pseudocomplementetd distributive lattice L is generalize Stone if and only if any two minimal prime ideals are comaximal.

Proof: Suppose L is generalized Stone. So any two minimal prime ideals are comaximal. To prove the converse, let P,Q be two minimal prime ideals of L.

We need to show that $[0,x]$ is stone,

For each $x \in L$ Let P_1, Q_1 be two minimal prime ideals in $[0,x]$. Using Lemma 2. there exists minimal prime Ideals P,Q in L such that

$P_1 = P \cap [0,x], Q_1 = Q \cap [0,x]$.

Therefore

$P_1 \vee Q_1 = (P \cap [0,x]) \vee (Q \cap [0,x])$
$= [P \vee Q] \cap [0,x] = L \cap [0,x] = [0,x]$.

Therefore $[0,x]$ is stone. So L is generalized stone. □

Corollary 1. A distributive lattice L is generalized Stone if and only if it is sectionally complemented and normal.

Figure 1. the lattice L is in fact a generalized Stone lattice, as it is both sectionally complemented and normal.

Corollary 2. A distributive lattice L with 0 is generalized Stone if and only if it is normal and sectionally complemented. The following theorem is generalization of [3]

Theorem 3. If L is a distributive sectionally pseudocomplemented lattice, then L_F is a distributive pseudocomplemented lattice.

Proof: Suppose L is sectionally pseudocomplemented. Since L_F is a distributive lattice. Let $[x]\in L_F$,

Then $[0]\subseteq[x]\subseteq F$. Now $0\le x\wedge f\le f$, for all $f\in F$.

Let y be the pseudocomplemented of $x\wedge f$ in $[0,f]$ then $y\wedge x\wedge f=0$ implies $[y\wedge f]\wedge[x]=[0]$,

that is $[y]\wedge[x]=[0]$.

Suppose $[z]\wedge[x]=[0]$,

for some $[z]\in L_F$ then $z\wedge x\equiv 0(\psi_F)$.

This implies $z\wedge x\wedge f'=0$...............(i)

For some $f''\in F$.

Since $z\equiv z\wedge f(\psi_F)$,

so $z\wedge f''=z\wedge f=\wedge f''$.........$(ii)$

for some $f''\in F$.

From (i) and (ii) we get $x\wedge x\wedge f'\wedge f''=0$

and $x\wedge f'\wedge f''=z\wedge x\wedge f'\wedge f''$

Setting $g=f'\wedge f''$ we have

$z\wedge g\wedge=z\wedge g\wedge f$, which

implies $z\wedge g\le f$ and $z\wedge g\wedge f=0$

So $0\le z\wedge g\le f$ and $z\wedge g\le y$.

Hence, $[z\wedge g]\subseteq[y]$ But $[z]=[z\wedge g]$

as $g\in F$. Therefore, $[z]\subseteq[y]$, and

so L_F is a pseudocomplemented distributive lattice. □

Theorem 4. Suppose L be a relatively pseudocomplemented lattice.

Let $x\le y\le z$ in L and l be the relative pseudocomplement of y in $[x,z]$. Then for any $r\in L$,

$l\wedge r$ is the relative pseudocomplement of $y\wedge r$ in $[x\wedge r,z\wedge r]$

Proof: Suppose $t\wedge r$ is the relative pseudocomplement of $y\wedge r$ in $[x\wedge r,z\wedge r]$ Since l is the relative pseudocomplement of y in $[x,z]$,

so $l\wedge y=x$.

Thus, $(l\wedge r)\wedge(y\wedge r)=x\wedge r$

This implies $l\wedge r\le t\wedge r$

Again, $x\le l\vee(t\wedge r)\le z$ and

$y\wedge(l\vee(t\wedge r))$
$=(y\wedge l)\vee((y\wedge r)\wedge(t\wedge r))$
$=x\vee(x\wedge r)$

implies $l\vee(t\wedge r)\le l$;

$[x\wedge r,z\wedge r]\le l$ i.e, $l=l\vee(t\wedge r)$

Hence $t\wedge r\le l$, and so $t\wedge r\le l\wedge r$.

This implies $t\wedge r=l\wedge r$

Therefore $l\wedge r$ is the relative pseudocomplement of $y\wedge r$ in $[x\wedge r,z\wedge r]$.
□

[4] extended the notion of pseudocomplementation for meet semilattices, following theorem generalizes.

Theorem 5. If L is a distributive relatively pseudocomplemented lattice, then L_F is a distributive relatively pseudocomplemented lattice.

Proof: Since L_F is a distributive lattice.

Let $[x],[y],[z] \in L_F$ with $[x] \subseteq [y] \subseteq [z]$.

Then $[x] = [x \wedge y]$ and $[y] = [y \wedge z]$

Thus, $y \equiv x \wedge y(\psi_F)$ and $y \equiv x \wedge y(\psi_F)$

This implies $x \wedge f = x \wedge y \wedge f$

and $y \wedge g = y \wedge z \wedge g$ for some $f, g \in F$ then $y \wedge f \wedge g = y \wedge z \wedge f \wedge g$,

and $y \wedge f \wedge g = y \wedge z \wedge f \wedge g$, and so

$x \wedge f \wedge g \leq y \wedge f \wedge g \leq z \wedge f \wedge g$,

that is $x \wedge h \leq y \wedge h \leq z \wedge g$

where $f \wedge g \in F$

Suppose t is the relative pseudocomplement of $y \wedge h$ in $[x \wedge h, z \wedge h]$. Then $t \wedge y \wedge h = x \wedge h$, and so $[t] \wedge [y \wedge h] = [x \wedge h]$.

That is, $[t] \wedge [y] = [x]$

as $y \equiv y \wedge h(\psi_F)$ $y \equiv y \wedge h(\psi_F)$

and $x \equiv x \wedge h(\psi_F)$ Moreover,

$[t] \wedge [z] = [t] \wedge [z \wedge h] = [t \wedge z \wedge h] = [t]$

implies $[x] \subseteq [t] \subseteq [z]$

We claim that $[t]$ is the relative pseudocomplement of $[y]$ in $[[x],[z]]$ in L_F

Suppose $[l] \wedge [y] = [x]$ for some $[l] \in [[x],[z]]$. Then $l \wedge y \equiv x(\psi_F)$ and so $l \wedge y \wedge f' = x \wedge f'$ for some $f' \in F$ Again $[l] \subseteq [z]$ implies $l \equiv l \wedge z(\psi_F)$, and so $l \wedge g' = l \wedge z \wedge g'$ for some $g' \in F$.

Then $l \wedge y \wedge f' \wedge g' = x \wedge f' \wedge g'$

and $l \wedge f' \wedge g' = l \wedge z \wedge f' \wedge g'$

Thus, $l \wedge k = l \wedge x \wedge k$ and

$l \wedge k = l \wedge z \wedge k$ where $k = f' \wedge g' \in F$

This implies $x \wedge h \wedge k \leq l \wedge h \wedge k \leq z \wedge h \wedge k$

and $(l \wedge h \wedge k) \wedge (y \wedge h \wedge k \leq z \wedge h \wedge k$

Then $[l \wedge h \wedge k \leq t \wedge k]$.

Hence $[l] = [l \wedge h \wedge k] \subseteq [t \wedge k] = [t]$

And so t is the relative pseudocomplement of $[y]$ in $[[x],[y]]$.

Therefore, L_F is relative pseudocomplemented. □

The following theorem is extension of [6, theorem 4.1]

Theorem 6. For a distributive sectionally pseudocomplemented lattice L, the following statements are hold:

(i) If L is generalized stone then L_F is Stone for any filter F of L.

(ii) L is generalized Stone if and only if for each prime filter F of L, L_F is dense lattice

Proof: (i) Let $\psi_F(x), \psi_F(y) \in L_F$ be such that $\psi_F(x) \wedge \psi_F(y) = 0$ Then, $x \wedge y \equiv 0(\psi_F)$, which implies that $x \wedge y \wedge f = 0$ for some $f \in F$. Since L is generalized stone, then L is normal, so $(x]^* \vee (y \wedge f]^* = L$ Hence

$(\Psi_F(x)]^* \vee (\Psi_F(y)]^*$
$= (\Psi_F(x)]^* \vee (\Psi_F(y \wedge f]^*$

$= \psi_F((x)]^* \vee (y \wedge f]^*) = \psi_F(L) = L_F$

Thus, L_F is normal.

Again, since L is sectionally pseudocomplemented, then L_F is pseudocomplemented, Hence L_F is stone.

(ii) Suppose L is generalized Stone. Let $\psi_F(x) \neq 0$ and $\psi_F(q) \in (\psi_F(x)]^*$. Then $\psi_F(q) \wedge \psi_F(x) = 0$. Then F is contained in a unique ultra filter of L.

Thus L_F has a unique ultra filter; and so L_F has a unique minimal prime ideal.

But the zero ideal of L_F (as $0 \in L$) is the intersection of all the minimal prime ideals of L_F. Hence, by uniqueness, it is (minimal) prime ideal of L_F. Hence $\psi_F(q) = 0$ showing that L_F is dense.

Conversely, let L_F be dense for each prime filter F of L.

Suppose $x, y \in L$ are such that

$x \wedge y = 0$ Then $\psi_F(x \wedge y) = \psi_F(0) = 0$

That is $\psi_F(x) \wedge \psi_F(y) = 0$ which implies that $\psi_F(x) = 0$ or $\psi_F(y) = 0$ as L_F is dense.

Hence, either $(\psi_F(x)]^* = L_F$

or $(\psi_F(y)]^* = L_F$.

Thus $\psi_F((x)] \vee (y]^*) = L_F = \psi_F(L)$

and so $(x]^* \vee (y]^* = L)$

Therefore L is normal.

Again, since L is sectionally pseudocomplemented, so L is generalized Stone. □

3. Conclusions

We have shown that a distributive lattice L with 0 is generalized Stone if and only if it is both normal and sectionally pseudocomplemented. In fact a generalized Stone lattice, as it is both sectionally complemented and normal.

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