EXACT SOLUTION FOR THE TZITZEICA-DODD-BULLOUGH EQUATION AND DODD-BULLOUGH-MIKHAILOV EQUATION BY TANH METHOD

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Abstract: *Analytical technique Tanh method is applied in this work for finding solution of two nonlinear partial differential equations (NLPDEs) named as Tzitzeica-Dodd-Bullough (TDB) equation and Dodd-Bullough-Mikhailov (DBM) equation which contain exponential terms.*

Keywords: *Tzitzeica-Dodd-Bullough (TDB), Dodd-Bullough-Mikhailov (DBM), Tanh, NLPDE.*

1. Introduction

The study of nonlinear physical phenomena ideally encompasses the exploration of exact solutions of nonlinear evolution equations (NLEEs) for their importance. Numerous effectual approaches and techniques such as Exp-function method [8], Tanh method [1-12], Sech method [1] etc. The Tanh method was introduced by Malfliet [12] in 1992. Later, this method was developed through several works[5,6].The tanh method is based on an ansatz that the traveling wave solutions can be expressed in terms of the tanh function to solve TDB equation and DBM equation.

The class of equations [4], namely, $u_{xt} + f(u) = 0$, where $f(u) \in \{ \sin u, \sinh u, e^{u}, pe^{u} + qe^{-2u}, e^{-u} + e^{-2u} \}$ (1) play a significant role in many scientific applications such as solid-state physics, nonlinear optics and quantum field theory.

A case of (1) is the equation with the function $f(u) = e^{-u} + e^{-2u}$ is known as the Tzitzeica-Dodd-Bullough (TDB) equation which is in our consideration. We also presented Dodd-Bullough-Mikhailov (DBM) equation involving the exponential function $f(u) = pe^{u} + qe^{-2u}$ another case of (1).

2. Solution Approach and Outline of Tanh Method

We consider the nonlinear partial differential equation in the form [1]

$$G(u, u_t, u_x, u_{xx}, u_x, u_{xt}, \dots \dots) = 0 \quad \dots (2)$$

where $u(x,t)$ is the solution of the partial differential equation (2).

Let's take the transformation, $u(x,t) = f(\eta)$ where $\eta = \alpha x + \beta t$ .... (3)

Then we have the following transformation of differential operators:

$$\frac{\partial}{\partial t}(.) = \beta \frac{d}{d\eta}(.), \frac{\partial}{\partial x}(.) = \alpha \frac{d}{d\eta}(.), \frac{\partial^2}{\partial x^2}(.) = \alpha^2 \frac{d^2}{d\eta^2}(.), \frac{\partial^2}{\partial t^2}(.) = \beta^2 \frac{d^2}{d\eta^2}(.) \dots\dots\dots (4)$$

Using (4) into equation (2), we get

$$G(f, f', f'', f''', f'''') = 0 \dots\dots\dots (5)$$

Equation (5) is an ODE and can be integrated as many times as all terms contain derivative and one can set integrating constant to zero since $u(\eta), \frac{du}{d\eta}, \frac{d^2u}{d\eta^2}, \frac{d^3u}{d\eta^3} \dots \dots \dots \to 0$ as $\eta \to 0$ for traveling wave.

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We now describe the tanh method for partial differential equation (2). To use this method, consider the following substitution: [7]

$Y = \tanh(\eta)$

that leads to the change of variable:

$$\frac{dV}{d\eta} = (1-Y^2)\frac{dV}{dY}$$

$$\frac{d^2V}{d\eta^2} = -2Y(1-Y^2)\frac{dV}{dY} + (1-Y^2)^2\frac{d^2V}{dY^2}$$

$$\frac{d^3V}{d\eta^3} = 2(1-Y^2)(3Y^2-1)$$

$$\frac{dV}{dY} - 6Y(1-Y^2)^2\frac{d^2V}{dY^2} + (1-Y^2)^3\frac{d^3V}{dY^3} \quad \ldots\ldots.(6)$$

The next crucial step is that the solution we are looking for is expressed in the form

$$V(\eta) = \sum_{i=0}^{m} a^i Y^i \quad \ldots\ldots\ldots\ldots\ldots (7)$$

where the parameters m can be found by balancing the heighest-order linear terms with the nonlinear terms in equation (5), and $\alpha, \beta, \eta, a_0, a_1 \ldots\ldots\ldots\ldots a_m$ are to be determined. Substituting (6) into (5) will yield a set of algebraic equation for $\alpha, \beta, \eta, a_0, a_1 \ldots\ldots\ldots\ldots a_m$ because all coefficients of Y^i have to vanish. From this relations $\alpha, \beta, \eta, a_0, a_1 \ldots\ldots\ldots\ldots a_m$ can be obtained. Having determined these parameters, knowing that m is positive integer in most cases, and using (7), we obtain analytic solution $V(\eta)$ in a closed form [7]. The tanh method seems to be powerful tool in dealing coupled nonlinear physical models.

3. Application of Tanh Method

Example 1:

Consider the Tzitzeica-Dodd-Bullough (TDB) equation [4]

$$u_{xt} + e^{-u} + e^{-2u} = 0 \ldots\ldots\ldots.(8)$$

This equation appears in problems varying from fluid flow to quantum field theory. For solving this equation and for finding major solutions, we use the transformations:

$$v(x,t) = e^{-u};\ u(x,t) = -\ln(v(x,t)) \ \ldots\ \ldots.(9)$$

Equation (8) becomes a partial deferential equation, which reads

$$-vv_{xt} + v_x v_t + v^3 + v^4 = 0 \ \ldots\ldots.(10)$$

To find the traveling wave solution of Equation (10), we introduce the wave variable

$$\eta = \alpha x + \beta t \ldots\ldots\ldots.(11)$$

so that the nonlinear partial differential equation (10) is transformed into an ordinary differential equation

$$-\alpha\beta VV'' + \alpha\beta(V')^2 + V^3 + V^4 = 0 \ldots\ldots (12)$$

Applying tanh method,

$$-\alpha\beta V[-2Y(1-Y^2)\frac{dV}{dY} + (1-Y^2)^2\frac{d^2V}{dY^2}]$$

$$+\alpha\beta(1-Y^2)^2(\frac{dV}{dY})^2 + V^3 + V^4 = 0 \ldots(13)$$

In this case, we consider equation (13) using tanh method. For determining value of m in equation (7), we balance the linear term of the highest order with the highest order nonlinear term in equation (13) that yields m = 1.

Therefore, we have:

$$V(\eta) = a_0 + a_1 Y \ldots\ldots\ldots.(14)$$

Applying tanh method with the transformation, equation (13) becomes:

$$-\alpha\beta(a_0 + a_1 Y)[2Y(1+Y^2)a_1] - \alpha\beta(1+Y^2)^2 a_1^2$$

$$+(a_0 + a_1 Y)^3 + (a_0 + a_1 Y)^4 = 0 \ldots\ldots(15)$$

Then equating the coefficient of $Y^i; i = 0,1,2$ from equation (15) leads to the following nonlinear system of algebraic equations:

$$Y^0 : -\alpha\beta a_1^2 + a_0^3 + a_0^4 = 0$$

$$Y^1 : -2\alpha\beta a_0 a_1 + 3a_0^2 a_1 + 4a_0^3 a_1 = 0$$

$$Y^2 : -2\alpha\beta a_1^2 - 2\alpha\beta a_1^2 + 3a_0 a_1^2 + 6a_0^2 a_1^2 = 0$$

Solving the nonlinear system of algebraic equations to get the following cases:

Case-1: $\alpha = \frac{9}{4\beta}; a_0 = -\frac{3}{2}; a_1 = \frac{\sqrt{3}}{2}$

Case-2: $\alpha = \frac{9}{4\beta}; a_0 = -\frac{3}{2}; a_1 = -\frac{\sqrt{3}}{2}$

Inserting these values into ansatz equation (14), we obtain:

$$V(\eta) = -\frac{1}{2} \pm \frac{1}{2}\tanh(\eta) \ldots\ldots\ldots.(16)$$

Substituting $\eta = \alpha(x + \frac{\beta}{\alpha}t)$ into this result, we obtain:

$$v(x,t) = -\frac{1}{2} + \frac{1}{2}\tanh\alpha(x - \frac{9}{4\alpha^2}t) \quad \ldots (17)$$

and $v(x,t) = -\frac{1}{2} - \frac{1}{2}\tanh\alpha(x - \frac{9}{4\alpha^2}t) \quad \ldots (18)$

We can obtain

$$u(x,t)\ u(x,t) = -\ln(-\frac{1}{2} + \frac{1}{2}\tanh\alpha(x - \frac{9}{4\alpha^2}t)) \quad \ldots (19)$$

and $u(x,t) = -\ln(-\frac{1}{2} - \frac{1}{2}\tanh\alpha(x - \frac{9}{4\alpha^2}t)) \quad (20)$

Example 2:

Consider the Dodd-Bullough-Mikhailov equation [7]

$$u_{xt} + pe^{u} + qe^{-2u} = 0 \ldots\ldots\ldots (21)$$

which is the Liouville equation if q=0. [4]

From the transformation, $u = \ln v, v(x,t) = V(\eta)$, equation (21) becomes a partial differential equation, which reads

$$vv_{xt} - v_x v_t + pv^3 + q = 0 \ \ldots (22)$$

To find the traveling wave solution of equation (22), we introduce the wave variable

$$\eta = \alpha x + \beta t \ldots\ldots\ldots (23)$$

so that the nonlinear partial differential equation (22) is carried to an ordinary differential equation:

$$\alpha\beta VV'' - \alpha\beta(V')^2 + pV^3 + q = 0 \ldots (24)$$

Applying tanh method,

$$\alpha\beta V[-2Y(1-Y^2)\frac{dV}{dY} - (1-Y^2)^2 \frac{d^2V}{dY^2}] + \alpha\beta(1-Y^2)^2(\frac{dV}{dY})^2 + pV^3 + q = 0 \quad \ldots\ldots(25)$$

In this case, we consider equation (25) using tanh method. To determine value m, we balance the linear term of the highest order with the highest order nonlinear term in equation (25) that yields m = 2.

Therefore, we have:

$$V(\eta) = a_0 + a_1 Y + a_2 Y^2 \ldots\ldots\ldots (26)$$

Applying tanh method with the transformation, equation (14) becomes

$$\alpha\beta(a_0 + a_1 Y + a_2 Y)[-2Y(1-Y^2)(a_1 + 2a_2 Y) - (1-Y^2)^2 2a_2] - \alpha\beta(1-Y^2)^2 a_1^{\,2} + p(a_0 + a_1 Y + a_2 Y)^3 + q = 0 \quad \ldots\ldots (27)$$

Then equating the coefficient of $Y^i; i = 0,1,2,3,\ldots$ in the equation (27) leads to the following nonlinear system of algebraic equations:

$$Y^0 : q + pa_0^3 - \alpha\beta a_1^2 + 2\alpha\beta a_0 a_2 = 0$$

$$Y^1 : -2\alpha\beta a_0 a_1 + 3pa_0^2 a_1 - 2\alpha\beta a_1 a_2 = 0$$

$$Y^2 : 3pa_0 a_1^2 - 8\alpha\beta a_0 a_2 + 3pa_0^2 a_2 - 2\alpha\beta a_2^2 = 0$$

$$Y^3 : 2\alpha\beta a_0 a_1 + pa_1^3 + 2\alpha\beta a_1 a_2 + 6pa_0 a_1 a_2 = 0$$

$$Y^4 : \alpha\beta a_1^2 + 6\alpha\beta a_0 a_2 + 3pa_1^2 a_2 + 3pa_0 a_2^2 = 0$$

$$Y^5 : 4\alpha\beta a_1 a_2 + 3pa_1 a_2^2 = 0$$

$$Y^6 : 2\alpha\beta a_2^2 + pa_2^3 = 0$$

Solving it by Mathematica gives two solutions:

$$q = 0, a_1 = 0, a_0 = -a_2, a_2 = \frac{-2\alpha\beta}{p} \ldots(28)$$

with a_2 being constant, and

$$a_1 = 0, a_0 = \frac{3q^{1/3}}{2p^{1/3}}, a_2 = -\frac{q^{1/3}}{2p^{1/3}}, \alpha\beta = \frac{(p^2 q)^{1/3}}{4} \quad \ldots\ldots(29)$$

For the Liouville equation $(q = 0)$, we obtain the solution:

$$u = \ln\left(-2\alpha\beta/p\, \mathrm{sech}^2(\alpha x + \beta t)\right)$$

We get the solution for the Dodd-Bullough-Mikhailov equation $(q \neq 0)$

$$u_1 = \ln\left[\frac{q^{1/3}}{2p^{1/3}}(3 - \tanh^2(\alpha x + \beta t))\right]$$

4. Conclusions

The tanh method is presented and the Tzitzeica-Dodd-Bullough (TDB) equation is solved using this method adopting some transformations and coefficient determination technique (comparing coefficients and solving system of equations for finding the values of undetermined coefficients). Thus, the approach of finding the exact solution of TDB is presented which explores the power of this method.

The tanh method is also used for finding the exact solution of the Dodd-Bullough-Mikhailov (DBM) equation in the same fashion. This method can easily be utilized to other nonlinear evolution equations of any order and the method can be approached to extended tanh method for solving different nonlinear differential equations. Any symbolic computation package (Mathematica, Matlab, Maple, etc.) can be of help in some cases involving tedious manual calculations.

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