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ANALYTICAL APPROXIMATE SOLUTION OF NONLINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract: In this paper, Shehu Transform Homotopy Analysis Method (STHAM) is proposed for the solution of nonlinear fractional order ordinary and partial differential equations. The interpretation of fractional order derivative is done in Caputo sense, while the nonlinearity encountered is overcome by exploiting the homotopy derivatives. The approach reduces the volume of computations unlike some other methods in the literature. The proposed method produces exact solution when such exists in closed form.

Keywords: *Shehu Transform, Caputo Derivative, Homotopy Derivative, Control Parameter, Embedding Parameter, Deformation Equations.*

1. INTRODUCTION

The attention of researchers has majorly shifted to finding reliable solutions to fractional order problems, be it ordinary differential equations, partial differential equations, integral or integro-differential equations. The reason for such shift in direction premised on the fact that modeling real situations in engineering, mathematical physics [1], mathematical biology [2], chemistry and other fields of sciences mostly resort to fractional calculus [3], and the need to develop more efficient methods cannot be overemphasized.

Since most physical phenomena are modeled into nonlinear ordinary differential equations, partial differential equations and integro-differential equations which are mostly defy solutions by known analytical methods, therefore integral transforms and analytical approximation methods are resorted into. Prominent among such methods are those reported in [4], [5], [6], [7], just to mention a few. Specific attention to the solution of problems on fractional order linear and nonlinear differential and integral equations include those of [8] - [16] and the references in them. [17] worked on the conversgence of one of the methods proposed in the previous works, while [18] worked on the generalization of the definition of the fractional derivatives.

The present work therefore proposed a more efficient method of solution that combines Laplace-type integral transform with homotopy analysis method for the solution of linear and nonlinear fractional order ordinary and partial differential equations.

2. STATEMENT OF THE PROBLEM

The family of the problems that are solved using the proposed method are as stated below.

2.1 ***Fractional Order Nonliear Ordinary Differential Equation***

$$D^{\alpha}u(x) + N\big(u(x)\big) + Qu(x) = f(x), \quad 0 < \alpha \leq \mu \in \mathbb{N}, \quad (2.1)$$

where $\boldsymbol{N(u(x))}$ is the nonlinear term, $\boldsymbol{Qu(x)}$ is the remaining linear term and $\boldsymbol{f(x)}$ is the inhomogeneous source term.

2.2 ***Fractional Order Nonlinear Partial Differential Equation***

$$\boldsymbol{D^{\alpha}u(x,t) + N\big(u(x,t)\big) + Qu(x,t) = f(x,t), \quad 0 < \alpha \leq \mu} \in \mathbb{N}, \quad (\boldsymbol{2.2})$$

where $\boldsymbol{N(u(x,t))}$ is the nonlinear term, $\boldsymbol{Qu(x,t)}$ is the remaining linear term and $\boldsymbol{f(x,t)}$ is the inhomogeneous source term.

3. METHODOLOGY

In this section, we present the algorithm of our proposed methods for the solutions of the problems stated in (2.1) and (2.2) above.

Some basic definitions and details such as Shehu transform, Homotopy analysis method, Riemann-Liouville integral and derivatives are not stated here since they are available in the already cited literatures. Nonetheless, the definition of Caputo derivative is stated for the reason discussed in the sequel.

Meanwhile, the choice of Shehu transform method was informed by the fact that it generalizes the two earlier transforms; the Laplace and Sumudu transforms. Its application is equally not restricted to constant coefficient problems, unlike Laplace transform [5].

3.1 Caputo Derivatives

The requirement of initial conditions by Riemann-Liouville fractional derivatives restricts it application to a wide range of practical problems as such conditions are mostly not readily available. Therefore, the most suitable definition for fractional derivatives is the Caputo fractional order derivative which does not make availability of initial conditions a prerequisite for its applicability. See [6] and the references therein.

Definition

Let $\eta \in \mathbb{R}_+$ and $\xi = \lceil \eta \rceil$. The operator

$$D_{*a}^{\eta} f(t) = J_a^{\xi-\eta} D^{\xi} f(t) = \frac{1}{\Gamma(\xi-\eta)} \int_a^t (t-\tau)^{\xi-\eta-1} \left(\frac{d}{d\tau}\right)^{\xi} f(\tau) d\tau,$$

for $a \le t \le b$, is the Caputo differential operator of order η.

This definition and the details of its proof are available in [2] for interested readers.

3.2 Algorithm for Nonlinear ODE

Consider the fractional order ordinary differential equation (ODE) in (2.1).

We apply Shehu transform to both sides of (2.1) to get

$$S\{D^{\alpha}u(x)\} + S\{N(u(x))\} + S\{Qu(x)\} = S\{f(x)\}, \quad (3.1)$$

But

$$S\{D^{\alpha}u(x)\} = \left(\frac{s}{u}\right)^{\alpha} U(s,u) - \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) \quad (3.2)$$

Using (3.2) in (3.1) yields

$$\left(\frac{s}{u}\right)^{\alpha} U(s,u) - \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) + S\{N(u(x))\} + S\{Qu(x)\} - S\{f(x)\} = 0$$

$$U(s,u) - \left(\frac{u}{s}\right)^{\alpha} \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) + \left(\frac{u}{s}\right)^{\alpha} S\{N(u(x))\} + \left(\frac{u}{s}\right)^{\alpha} S\{Qu(x)\} - \left(\frac{u}{s}\right)^{\alpha} S\{f(x)\} = 0 \quad (3.3)$$

The jth order deformation equation is given as

$$L[U_j((s,u);\eta) - \chi_j U_{j-1}((s,u);\eta)] = \xi D_{j-1}[N[\phi(x;\eta)]], \quad (3.4)$$

Where η is the embedding parameter, ξ is the control parameter, D_{j-1} is the $(j-1)th$ order homotopy derivative and $\chi_j = \begin{cases} 0, & j \le 0 \\ 1, & j > 0 \end{cases}$

The general nonlinear operator is derived from (3.3) as

$$N[\phi(x;\eta)] = U(s,u) - \left(\frac{u}{s}\right)^{\alpha} \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) + \left(\frac{u}{s}\right)^{\alpha} S\{N(u(x))\} + \left(\frac{u}{s}\right)^{\alpha} S\{Qu(x)\} - \left(\frac{u}{s}\right)^{\alpha} S\{f(x)\} \quad (3.5)$$

Also, the auxiliary linear operator gives

$$L[U_j(s,u);\eta] = U_j(s,u) \quad (3.6)$$

Using (3.5) and (3.6) in (3.5) gives

$$U_j(s,u) - \chi_j U_{j-1}(s,u) = \xi\, \mathrm{D}_{j-1}\left(\mathrm{U}(\mathrm{s},\mathrm{u}) - \left(\frac{u}{s}\right)^{\alpha} \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) + \left(\frac{u}{s}\right)^{\alpha} S\{N(u(x))\} + \left(\frac{u}{s}\right)^{\alpha} \mathrm{S}\{\mathrm{Qu}(x)\} - \left(\frac{u}{s}\right)^{\alpha} \mathrm{S}\{\mathrm{f}(\mathrm{x})\} \right)$$

$$U_j(s,u) - \chi_j U_{j-1}(s,u) = \xi\left(U_{j-1}(s,\mathrm{u}) - (1-\chi_{j-1}) \left(\frac{u}{s}\right)^{\alpha} \left[S\{f(x)\} + \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) \right] + \left(\frac{u}{s}\right)^{\alpha} \mathrm{S}\{D_{j-1}[N(u(x))]\} + \left(\frac{u}{s}\right)^{\alpha} \mathrm{S}\{D_{j-1}\mathrm{Q}[(x)]\} \right)$$

(3.7)

Let $\xi = -1$, (3.7) becomes

$$U_j(s,u) = -(1-\chi_j) U_{j-1}(s,u) U_{j-1}(s,\mathrm{u}) + (1-\chi_{j-1}) \left(\frac{u}{s}\right)^{\alpha} \left[S\{f(x)\} + \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} u^{(i)}(0) \right] - \left(\frac{u}{s}\right)^{\alpha} \mathrm{S}\{D_{j-1}[N(u(x))]\} - \left(\frac{u}{s}\right)^{\alpha} \mathrm{S}\{D_{j-1}\mathrm{Q}[(x)]\} \quad (3.8)$$

where $\chi_{j-1} = \begin{cases} 0, & \mathrm{j}-1 < 1 \\ 1, & \mathrm{j}-1 \ge 1 \end{cases}$

The initial approximation $u_0(x)$ is derived from the initial condition, while the other terms $u_1(x)$, $u_2(x)$, etc are obtained through (3.8) with the inverse Shehu transform taken at the required points.

3.3 Algorithm for Nonlinear PDE

Consider the fractional order partial differential equation (PDE) in (2.2).

Shehu transform is applied to (2.2) as follows

$$S\{D^{\alpha}u(x,t)\} + S\{N(u(x,t))\} + S\{Qu(x,t)\} = S\{f(x,t)\} \quad (3.9)$$

The first term on the left-hand side of (3.9), using the Shehu transform for derivative, is obtained as

$$S\{D^{\alpha}u(x,t)\} = \left(\frac{s}{u}\right)^{\alpha} U((s,u),t) - \sum_{i=0}^{\alpha-1} \left(\frac{s}{u}\right)^{\alpha-1-i} \frac{\partial^i u(0,t)}{\partial x^i} \quad (3.10)$$

Using (3.10) in (3.9), we have

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$$U((s,u),t)-\left(\frac{u}{s}\right)^{\alpha}\sum_{i=0}^{\alpha-1}\left(\frac{s}{u}\right)^{\alpha-1-i}\frac{\partial^i u(0,t)}{\partial x^i}+\left(\frac{u}{s}\right)^{\alpha}S\{N(u(x,t))\}+\left(\frac{u}{s}\right)^{\alpha}S\{Qu(x,t)\}-\left(\frac{u}{s}\right)^{\alpha}S\{f(x,t)\}=0, \quad (3.11)$$

where $U((s,u),t)$ is the Shehu transform of the function $u(x,t)$ with reference to the independent variable x.

The jth order deformation equation which is derived from the zeroth order deformation equation after differentiating it j times with respect to the embedding parameter η and setting η to zero, is given as

$$L[U_j((s,u),t;\eta)-\chi_j U_{j-1}((s,u),t;\eta)]=\xi D_{j-1}[N[\phi(x,t;\eta)]], \quad (3.12)$$

where ξ is the control parameter, and $\chi_j=\begin{cases}0, & j\le 1\\ 1, & j>1\end{cases}$.

But $L[U((s,u),t;\eta)]=U_j((s,u),t), \quad (3.13)$

and the general nonlinear operator is obtained from (3.11) as

$$N[U((s,u),t;\eta)]=U((s,u),t)-\left(\frac{u}{s}\right)^{\alpha}\sum_{i=0}^{\alpha-1}\left(\frac{s}{u}\right)^{\alpha-1-i}\frac{\partial^i u(0,t)}{\partial x^i}+\left(\frac{u}{s}\right)^{\alpha}S\{N(u(x,t))\}+\left(\frac{u}{s}\right)^{\alpha}S\{Qu(x,t)\}-\left(\frac{u}{s}\right)^{\alpha}S\{f(x,t)\}. \quad (3.14)$$

Now, we use (3.13) and (3.14) in (3.12) to get

$$U_j((s,u),t)-\chi_j U_{j-1}((s,u),t)=\xi D_{j-1}\left(U((s,u),t)-\left(\frac{u}{s}\right)^{\alpha}\sum_{i=0}^{\alpha-1}\left(\frac{s}{u}\right)^{\alpha-1-i}\frac{\partial^i u(0,t)}{\partial x^i}+\left(\frac{u}{s}\right)^{\alpha}S\{N(u(x,t))\}+\left(\frac{u}{s}\right)^{\alpha}S\{Qu(x,t)\}-\left(\frac{u}{s}\right)^{\alpha}S\{f(x,t)\}\right) \quad (3.15)$$

Equation (3.15) now gives

$$U_j((s,u),t)-\chi_j U_{j-1}((s,u),t)=\xi\Bigg(U_{j-1}((s,u),t)-(1-\chi_{j-1})\left(\frac{u}{s}\right)^{\alpha}\left[S\{f(x,t)\}+\sum_{i=0}^{\alpha-1}\left(\frac{s}{u}\right)^{\alpha-1-i}\frac{\partial^i u(0,t)}{\partial x^i}\right]+\left(\frac{u}{s}\right)^{\alpha}D_{j-1}[S\{N(u(x,t))\}+S\{Qu(x,t)\}]\Bigg), (3.16)$$

where $\chi_{j-1}=\begin{cases}0, & j-1<1\\ 1, & j-1\ge 1\end{cases}$ and ξ is the control parameter.

If $\xi=-1$, (3.16) becomes

$$U_j((s,u),t)=-(1-\chi_j)U_{j-1}((s,u),t)+(1-\chi_{j-1})\left(\frac{u}{s}\right)^{\alpha}\left[S\{f(x,t)\}+\sum_{i=0}^{\alpha-1}\left(\frac{s}{u}\right)^{\alpha-1-i}\frac{\partial^i u(0,t)}{\partial x^i}\right]-\left(\frac{u}{s}\right)^{\alpha}D_{j-1}[S\{N(u(x,t))\}+S\{Qu(x,t)\}], (3.17)$$

Taking inverse Shehu transform of both sides of (3.17) for various values of j will give the solution when these individual results are summed up.

4. EXAMPLES ON ODE AND PDE

In this section, we present examples on the two algorithms discussed in the preceding sections.

4.1 Examples on Ordinary Differential Equations

Problem 1 [3]

Consider the nonlinear fractional order ODE below using Shehu transform homotopy analysis method

$$D^{\alpha}u(x)=1+u^2(x), \quad u(0)=0, \quad 0<\alpha\le 1 \ (i)$$

Solution

Taking the Shehu transform of both sides of (*i*), we have

$$S\{D^{\alpha}u(x)\}=S\{1\}+S\{u^2(x)\} \quad (ii)$$

But

$$S\{D^{\alpha}u(x)\}=\left(\frac{s}{u}\right)^{\alpha}U(s,u)-\left(\frac{s}{u}\right)^{\alpha-1}u(0), \quad (iii)$$

which reduces, upon implementation of the initial condition, to

$$S\{D^{\alpha}u(x)\}=\left(\frac{s}{u}\right)^{\alpha}U(s,u) \quad (iv)$$

Using (*iv*) in (*ii*), we get

$$U(s,u)-\left(\frac{u}{s}\right)^{\alpha+1}+\left(\frac{u}{s}\right)^{\alpha}S\{u^2(x)\}=0 \quad (v)$$

From (v), the general nonlinear term is obtained as

$$N[U_j((s,u);\eta)] = U(s,u) - \left(\frac{u}{s}\right)^{\alpha+1} + \left(\frac{u}{s}\right)^{\alpha} S\{u^2(x)\} \quad (vi)$$

Consider the jth order deformation equation

$$L[U_j((s,u);\eta) - \chi_j U_{j-1}((s,u);\eta)] = \xi D_{j-1}\left[N[U_j((s,u);\eta)]\right] \ (vii)$$

where η is the embedding parameter, ξ is the control parameter, and $\chi_j = \begin{cases} 0, & j \le 1 \\ 1, & j > 1 \end{cases}$.

But $L[U_j((s,u);\eta)] = U_j(s,u)$. $\quad (viii)$

Using (vi) and ($viii$) in (vii), we have

$$U_j(s,u) - \chi_j U_{j-1}(s,u) = \xi D_{j-1}\left(U(s,u) - \left(\frac{u}{s}\right)^{\alpha+1} - \left(\frac{u}{s}\right)^{\alpha} S u^2(x)\right)$$

$$U_j(s,u) - \chi_j U_{j-1}(s,u) = \xi\left(U_{j-1}(s,u) - (1-\chi_{j-1})\left(\frac{u}{s}\right)^{\alpha+1} - \left(\frac{u}{s}\right)^{\alpha} S\left\{D_{j-1}\{u^2\}\right\}\right)$$

With $\xi = -1$, we have

$$U_j(s,u) = -(1-\chi_j)U_{j-1}(s,u) + (1-\chi_{j-1})\left(\frac{u}{s}\right)^{\alpha+1} + \left(\frac{u}{s}\right)^{\alpha} S\left\{\sum_{i=0}^{j-1} u_{j-1-i}u_i\right\},$$

where $\chi_{j-1} = \begin{cases} 0, & j-1 < 1 \\ 1, & j-1 \ge 1 \end{cases}$.

The initial approximation $u_0(x)$ is obtained from the given boundary condition as

$$u_0(x) = 0.$$

$$U_1(s,u) = -(1-\chi_1)U_0(s,u) + (1-\chi_0)\left(\frac{u}{s}\right)^{\alpha+1} + \left(\frac{u}{s}\right)^{\alpha} S\{u_0u_0\}$$

$$U_1(s,u) = \left(\frac{u}{s}\right)^{\alpha+1}$$

Taking the inverse Shehu transform of both sides, we have

$$S^{-1}\{U_1(s,u)\} = S^{-1}\left\{\left(\frac{u}{s}\right)^{\alpha+1}\right\}$$

$$u_1(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}.$$

For $j = 2$:

$$U_2(s,u) = -(1-\chi_2)U_1(s,u) + (1-\chi_1)\left(\frac{u}{s}\right)^{\alpha+1} + \left(\frac{u}{s}\right)^{\alpha} S\{2u_1u_0\}$$

$$U_2(s,u) = 0.\left(\frac{u}{s}\right)^{\alpha+1} + \left(\frac{u}{s}\right)^{\alpha} S\left\{2.0.\frac{x^\alpha}{\Gamma(\alpha+1)}\right\} = 0$$

Taking the inverse Shehu transform of both sides yields

$$u_2(x) = 0.$$

For $j = 3$:

$$U_3(s,u) = \left(\frac{u}{s}\right)^{\alpha} S\{u_1,u_1\} = \left(\frac{u}{s}\right)^{\alpha} S\left\{\frac{x^{2\alpha}}{\Gamma^2(\alpha+1)}\right\}$$

$$U_3(s,u) = \left(\frac{u}{s}\right)^{\alpha} \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\left(\frac{u}{s}\right)^{2\alpha+1}$$

$$U_3(s,u) = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\left(\frac{u}{s}\right)^{3\alpha+1}$$

Taking the inverse Shehu transform of both sides, gives

$$u_3(x) = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}.$$

For $j = 4$:

$$u_4(x) = 0.$$

For $j = 5$:

$$u_5(x) = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}.$$

Solution to the given problem is therefore obtained as

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots$$

$$u(x) = 0 + \frac{x^\alpha}{\Gamma(\alpha+1)} + 0 + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + 0 + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\frac{x^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots$$

$$u(x) = \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\frac{x^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots$$

For $\alpha = 1$:

$$u(x) = \frac{x}{\Gamma(2)} + \frac{\Gamma(3)x^3}{\Gamma^2(2)\Gamma(4)} + \frac{2\Gamma(3)\Gamma(5)x^5}{\Gamma(2)\Gamma^2(2)\Gamma(4)\Gamma(6)} + ..$$

$$u(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Problem 2 [3]

Solve the nonlinear fractional order ordinary differential equation below by STHAM

$$D^\alpha u(t) = t + u^2(x), \quad 1 < \alpha \le 2, u(0) = 0, u'(0) = 1 \ (i)$$

Solution

Taking the Shehu transform of both sides of (i) gives

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$$S\{D^\alpha u(t)\} = S\{t\} + S\{u^2(x)\} \quad (ii)$$

$$\left(\frac{s}{u}\right)^\alpha U(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} u(0) - \left(\frac{s}{u}\right)^{\alpha-2} u'(0) = \left(\frac{u}{s}\right)^2 + S\{u^2(t)\}$$

$$U(s,u) - \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^{\alpha+2} - \left(\frac{u}{s}\right)^\alpha S\{u^2(t)\} = 0 \quad (iii)$$

From (*iii*), the general nonlinear term is obtained as

$$N[U_j((s,u);\eta)] = U(s,u) - \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^{\alpha+2} - \left(\frac{u}{s}\right)^\alpha S\{u^2(t)\} \quad (iv)$$

Also

$$L[U_j((s,u);\eta)] = U_j(s,u). \quad (v)$$

Thus, the jth order deformation equation

$$L[U_j((s,u);\eta) - \chi_j U_{j-1}((s,u);\eta)] = \xi D_{j-1}\left[N[U_j((s,u);\eta)]\right] \quad (vi)$$

becomes

$$U_j(s,u) - \chi_j U_{j-1}(s,u) = \xi D_{j-1}\left(U(s,u) - \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^{\alpha+2} - \left(\frac{u}{s}\right)^\alpha S\{u^2(x)\}\right) \quad (vii)$$

$$U_j(s,u) - \chi_j U_{j-1}(s,u) = \xi\left(U_{j-1}(s,u) - (1-\chi_{j-1})\left(\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^{\alpha+2}\right) - \left(\frac{u}{s}\right)^\alpha S\left\{\sum_{i=0}^{j-1} u_{j-1-i}u_i\right\}\right) \quad (viii)$$

where χ_j and χ_{j-1} have their usual meaning.

Let $\xi = -1$, so that

$$U_j(s,u) = -(1-\chi_j)U_{j-1}(s,u) + (1-\chi_{j-1})\left(\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^{\alpha+2}\right) + \left(\frac{u}{s}\right)^\alpha S\left\{\sum_{i=0}^{j-1} u_{j-1-i}u_i\right\} \quad (ix)$$

The initial approximation is

$$u_0(t) = u(0) + tu'(t) = t \quad (x)$$

For subsequent approximations, (*x*) is used in (*ix*) for various values of j.

For $j = 1, 2, 3, \ldots$

$$u_1(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}$$

$$u_2(t) = \frac{\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{4\Gamma(\alpha+4)2t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}$$

$$u_3(t) = \frac{4\Gamma(\alpha+3)\Gamma(2\alpha+4)t^{3\alpha+3}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{8\Gamma(\alpha+3)\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)\Gamma(3\alpha+5)} + \frac{\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma^2(\alpha+2)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+4)t^{3\alpha+3}}{\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(3\alpha+4)} + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma^2(\alpha+3)\Gamma(3\alpha+5)}.$$

$$u(t) = t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{4\Gamma(\alpha+4)2t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{4\Gamma(\alpha+3)\Gamma(2\alpha+4)t^{3\alpha+3}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)\Gamma(3\alpha+4)} + \frac{8\Gamma(\alpha+3)\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)\Gamma(3\alpha+5)} + \frac{\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma^2(\alpha+2)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+4)t^{3\alpha+3}}{\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(3\alpha+4)} + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma^2(\alpha+3)\Gamma(3\alpha+5)} + \cdots$$

When $\alpha = 2$:

$$u(t) = t + \frac{t^3}{6} + \frac{t^4}{12} + \frac{t^6}{90} + \frac{t^7}{252} + \frac{t^8}{2016} + \frac{t^9}{1440} + \ldots$$

4.2 EXAMPLES NONLINEAR PDE

Problem 1 [14]

Consider the nonlinear fractional order PDE

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} - u\frac{\partial^3 u}{\partial t^3} + u\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} = 0 \quad (i)$$

subject to $u(x,0) = \frac{8}{3}e^{\frac{x}{2}}, \quad t > 0, \quad 0 < \alpha \le 1.$

Solution

Taking the Shehu transform of (*i*), we have

$$S\left\{\frac{\partial^\alpha u}{\partial t^\alpha}\right\} - S\left\{\frac{\partial^3 u}{\partial x^2 \partial t}\right\} + S\left\{\frac{\partial u}{\partial x}\right\} - S\left\{u\frac{\partial^3 u}{\partial t^3}\right\} + S\left\{u\frac{\partial u}{\partial x}\right\} - S\left\{3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right\} = 0 \quad (ii)$$

But

$$S\left\{\frac{\partial^\alpha u}{\partial t^\alpha}\right\} = \left(\frac{s}{u}\right)^\alpha U(x,(s,u)) - \left(\frac{s}{u}\right)^{\alpha-1} u(x,0)$$

$$S\left\{\frac{\partial^\alpha u}{\partial t^\alpha}\right\} = \left(\frac{s}{u}\right)^\alpha U(x,(s,u)) - \frac{8}{3}e^{\frac{x}{2}}\left(\frac{s}{u}\right)^{\alpha-1} \qquad (iii)$$

Using (*iii*) in (*ii*), we get

$$U(x,(s,u)) - \frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial^3 u}{\partial x^2 \partial t}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial u}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial^3 u}{\partial t^3}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial u}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right\} = 0 \qquad (iv)$$

From (*iv*), the general nonlinear operator is obtained as

$$N[(x,(s,u);\eta)] = U(x,(s,u)) - \frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial^3 u}{\partial x^2 \partial t}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial u}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial^3 u}{\partial t^3}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial u}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right\} \qquad (v)$$

Also, the linear operator

$$L[U_j(x,(s,u);\eta)] = U_j(x,(s,u)) \qquad (vi)$$

The jth order deformation equation is

$$L[U_j(x,(s,u);\eta) - \chi_j U_{j-1}(x,(s,u);\eta)] = \xi D_{j-1}[N[\phi(x,(s,u);\eta)]] \qquad (vii)$$

Using (*v*) and (*vi*) in (*vii*), we get

$$U_j(x,(s,u)) - \chi_j U_{j-1}(x,(s,u)) = \xi D_{j-1}\left(U(x,(s,u)) - \frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial^3 u}{\partial x^2 \partial t}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial u}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial^3 u}{\partial t^3}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial u}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right\}\right)$$

$$U_j(x,(s,u)) - \chi_j U_{j-1}(x,(s,u)) = \xi\left(U_{j-1}(x,(s,u)) - (1-\chi_{j-1})\frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial^3 u_{j-1}}{\partial x^2 \partial t}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial u_{j-1}}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial^3 u_{j-1}}{\partial t^3}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{\sum_{i=0}^{j-1} u_i \frac{\partial u_{j-1-i}}{\partial x}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{3\sum_{i=0}^{j-1}\frac{\partial u_i}{\partial x}\frac{\partial^2 u_{j-1-i}}{\partial x^2}\right\}\right) \qquad (viii)$$

Setting the control parameter, $\xi = -1$, (*viii*) becomes

$$U_j(x,(s,u)) = -(1-\chi_j)U_{j-1}(x,(s,u)) + (1-\chi_{j-1})\frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial^3 u_{j-1}}{\partial x^2 \partial t}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial u_{j-1}}{\partial x}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial^3 u_{j-1}}{\partial t^3}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{\sum_{i=0}^{j-1} u_i \frac{\partial u_{j-1-i}}{\partial x}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{3\sum_{i=0}^{j-1}\frac{\partial u_i}{\partial x}\frac{\partial^2 u_{j-1-i}}{\partial x^2}\right\} \qquad (ix)$$

The initial approximation is derived from the initial condition as

$$u_0(x,t) = u(x,0) = \frac{8}{3}e^{\frac{x}{2}}.$$

The Shehu transform of the initial approximation is

$$S\{u_0(x,t)\} = U_0(x,(s,u)) = S\left\{\frac{8}{3}e^{\frac{x}{2}}\right\} = \frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) \qquad (xi)$$

Using (*xi*) in (*ix*) for $j = 1, 2, 3, \ldots$, we get

$$U_1(x,(s,u)) = -(1-\chi_1)U_0(x,(s,u)) + (1-\chi_0)\frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial^3 u_0}{\partial x^2 \partial t}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{\partial u_0}{\partial x}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{u\frac{\partial^3 u_0}{\partial t^3}\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{u_0\frac{\partial u_0}{\partial x}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\right\}$$

$$U_1(x,(s,u)) = -\frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) + \frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^\alpha(0) - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{4}{3}e^{\frac{x}{2}}\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{\frac{8}{9}e^x\right\} - \left(\frac{u}{s}\right)^\alpha S\left\{\frac{32}{9}e^x\right\} + \left(\frac{u}{s}\right)^\alpha S\left\{3.\frac{8}{9}e^x\right\}$$

$$U_1(x,(s,u)) = -\left(\frac{u}{s}\right)^{\alpha+1}\frac{4}{3}e^{\frac{x}{2}} + \left(\frac{u}{s}\right)^{\alpha+1}\frac{8}{9}e^x - \left(\frac{u}{s}\right)^{\alpha+1}\frac{32}{9}e^x + \left(\frac{u}{s}\right)^{\alpha+1}\frac{8}{3}e^x.$$

$$U_1(x,(s,u)) = -\frac{4}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right)^{\alpha+1}$$

Taking inverse Shehu transform of both sides, we get

$$u_1(x,t) = -\frac{4}{3}e^{\frac{x}{2}}\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

For $j = 2$:

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$$\begin{aligned} U_2(x,(s,u)) = &-(1-\chi_2)U_1(x,(s,u)) \\ &+(1-\chi_1)\frac{8}{3}e^{\frac{x}{2}}\left(\frac{u}{s}\right) \\ &+\left(\frac{u}{s}\right)^{\alpha} S\left\{\frac{\partial^3 u_1}{\partial x^2 \partial t}\right\} \\ &-\left(\frac{u}{s}\right)^{\alpha} S\left\{\frac{\partial u_1}{\partial x}\right\} \\ &+\left(\frac{u}{s}\right)^{\alpha} S\left\{u\frac{\partial^3 u_1}{\partial t^3}\right\} \\ &-\left(\frac{u}{s}\right)^{\alpha} S\left\{u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}\right\} \\ &+\left(\frac{u}{s}\right)^{\alpha} 3S\left\{\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2} \right. \\ &\left. +\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\right\} \end{aligned}$$

$$\begin{aligned} U_2(x,(s,u)) = &-\left(\frac{u}{s}\right)^{\alpha} S\left\{-\frac{2}{3}e^{\frac{x}{2}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\} \\ &+\left(\frac{u}{s}\right)^{\alpha} S\left\{-\frac{1}{3}e^{\frac{x}{2}}\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\right\} \\ &+\left(\frac{u}{s}\right)^{\alpha} S\left\{-\frac{8}{9}e^{x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\} \\ &-\left(\frac{u}{s}\right)^{\alpha} S\left\{-\frac{32}{9}e^{x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\} \\ &+\left(\frac{u}{s}\right)^{\alpha} S\left\{-\frac{8}{3}e^{x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\}. \end{aligned}$$

$$\begin{aligned} U_2(x,(s,u)) = &\left(\frac{u}{s}\right)^{\alpha}\frac{2}{3}e^{\frac{x}{2}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left(\frac{u}{s}\right)^{\alpha+1} \\ &-\left(\frac{u}{s}\right)^{\alpha}\frac{1}{3}e^{\frac{x}{2}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left(\frac{u}{s}\right)^{\alpha} \end{aligned}$$

$$U_2(x,(s,u)) = \left(\frac{u}{s}\right)^{2\alpha+1}\frac{2}{3}e^{\frac{x}{2}}-\left(\frac{u}{s}\right)^{2\alpha}\frac{1}{3}e^{\frac{x}{2}}$$

Taking inverse Shehu transform of both sides, we get

$$u_2(x,t)=\frac{2}{3}e^{\frac{x}{2}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{1}{3}e^{\frac{x}{2}}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}.$$

The solution is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots$$

$$\begin{aligned} u(x,t)=&\frac{8}{3}e^{\frac{x}{2}}-\frac{4}{3}e^{\frac{x}{2}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2}{3}e^{\frac{x}{2}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &-\frac{1}{3}e^{\frac{x}{2}}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots \end{aligned}$$

Problem 2 [6]

Consider the one-dimensional linear fractional diffusion equation

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}=\frac{\partial^2 v(x,t)}{\partial x^2}+v(x,t), \qquad (i)$$

subject to the initial condition

$$v(x,0)=\cos(\pi x), \qquad 0<\alpha\le 1.$$

Solution

Applying Shehu transform to both sides of (*i*), we have

$$S\left\{\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}\right\}=S\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}+S\{v(x,t)\} \qquad (ii)$$

But

$$S\left\{\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}\right\}=\left(\frac{s}{v}\right)^{\alpha} V(x,(s,v))-\left(\frac{s}{v}\right)^{\alpha-1} v(x,0)$$

$$\begin{aligned} S\left\{\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}\right\}=&\left(\frac{s}{v}\right)^{\alpha} V(x,(s,v)) \\ &-\left(\frac{s}{v}\right)^{\alpha-1}\cos(\pi x) \end{aligned}$$

With this, (*ii*) becomes

$$\begin{aligned} V(x,(s,v))-\left(\frac{v}{s}\right)\cos(\pi x)-\left(\frac{v}{s}\right)^{\alpha} S\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} \\ -\left(\frac{v}{s}\right)^{\alpha} S\{v(x,t)\}=0 \qquad (iii) \end{aligned}$$

From (*iii*), the general nonlinear term is

$$N[\phi(x,(s,v);\eta)]=V(x,(s,v))-\left(\frac{v}{s}\right)\cos(\pi x)-\left(\frac{v}{s}\right)^{\alpha} S\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}-\left(\frac{v}{s}\right)^{\alpha} S\{v(x,t)\} \qquad (iv)$$

while the linear term is

$$\mathrm{L}[V_j(x,(s,v);\eta)]=V_j(x,(s,v)) \qquad (v)$$

The jth order deformation equation is given as

$$\begin{aligned} L[V_j(x,(s,v);\eta)-\chi_j V_{j-1}(x,(s,v);\eta)] \\ =\xi D_{j-1}[N[\phi(x,(s,v);\eta)]] \quad (vi) \end{aligned}$$

Using (*iv*) and (*v*) in (*vi*), we have

$$\begin{aligned} &V_j(x,(s,v))-\chi_j V_{j-1}(x,(s,v)) \\ &=\xi D_{j-1}\left(V(x,(s,v))-\left(\frac{v}{s}\right)\cos(\pi x)\right. \\ &-\left(\frac{v}{s}\right)^{\alpha} S\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\} \\ &\left.-\left(\frac{v}{s}\right)^{\alpha} S\{v(x,t)\}\right) \qquad (vii) \end{aligned}$$

$$\begin{aligned} V_j(x,(s,v))-\chi_j V_{j-1}(x,(s,v)) = \\ \xi\left(V_{j-1}(x,(s,v))-(1-\chi_{j-1})\left(\frac{v}{s}\right)\cos(\pi x)-\right. \\ \left.\left(\frac{v}{s}\right)^{\alpha} S\left\{\frac{\partial^2 u_{j-1}(x,t)}{\partial x^2}\right\}-\left(\frac{v}{s}\right)^{\alpha} S\{V_{j-1}(x,(s,v))\}\right) \quad (viii) \end{aligned}$$

χ_j and χ_{j-1} have the usual meaning.

Suppose the control parameter $\xi=-1$, (*viii*) becomes

$$\begin{aligned} V_j(x,(s,v))=&-(1-\chi_j)V_{j-1}(x,(s,v)) \\ &+(1-\chi_{j-1})\left(\frac{v}{s}\right)\cos(\pi x) \\ &+\left(\frac{v}{s}\right)^{\alpha} S\left\{\frac{\partial^2 u_{j-1}(x,t)}{\partial x^2}\right\} \\ &+\left(\frac{v}{s}\right)^{\alpha} S\{V_{j-1}(x,(s,v))\} \quad (ix) \end{aligned}$$

The initial approximation

$$v_0(x,t)=v(x,0)=cos(\pi x).$$

Other terms are obtained from (*ix*) for various values of j as follows:

For $j=1$:

$$V_1(x,(s,v))=\left(\frac{v}{s}\right)^{\alpha+1}[-\pi^2 cos(\pi x)+cos(\pi x)]$$

Taking the inverse Shehu transform of both sides, we have

$$v_1(x,t) = [cos(\pi x) - \pi^2 cos(\pi x)] \frac{t^\alpha}{\Gamma(\alpha+1)}$$

For $j = 2$:

$$V_2(x,(s,v)) = \left(\frac{v}{s}\right)^{2\alpha+1} [-2\pi^2 cos(\pi x) + cos(\pi x) + \pi^4 cos(\pi x)]$$

Taking the inverse Shehu transform yields

$$v_2(x,t) = [-2\pi^2 cos(\pi x) + cos(\pi x) + \pi^4 cos(\pi x)] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

For $j = 3$:

$$V_3(x,(s,v)) = \left(\frac{v}{s}\right)^{3\alpha+1} [3\pi^4 cos(\pi x) - 3\pi^2 cos(\pi x) + cos(\pi x) - \pi^6 cos(\pi x)]$$

Taking inverse Shehu transform, we get

$$v_3(x,t) = \Big[[3\pi^4 cos(\pi x) - 3\pi^2 cos(\pi x) + cos(\pi x) - \pi^6 cos(\pi x)]\Big] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

From the foregoing, the final answer shall be obtained as

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + ..$$

Thus,

$$v(x,t) = cos(\pi x) + [cos(\pi x) - \pi^2 cos(\pi x)] \frac{t^\alpha}{\Gamma(\alpha+1)} + [-2\pi^2 cos(\pi x) + cos(\pi x) + \pi^4 cos(\pi x)] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \Big[[3\pi^4 cos(\pi x) - 3\pi^2 cos(\pi x) + cos(\pi x) - \pi^6 cos(\pi x)]\Big] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots$$

5. Discussion of Results

The proposed method, STHAM has been applied to nonlinear fractional order ordinary differential equations as well as linear and nonlinear fractional order partial differential equations that are homogeneous and inhomogeneous problems. The solutions obtained for the selected problems from the literatures tally with the solutions obtained through other methods in the literature, at reduced computational time and space. In all the nonlinear problems considered, the nonlinearities are resolved with the aid of the concept of homotopy derivatives.

6. Conclusion

The proposed semi analytical method STHAM has been developed and successfully applied to selected nonlinear fractional order ODEs and PDEs. Homotopy derivative has equally been deployed to overcome the nonlinearities encountered in all cases.

In the nearest future, the STHAM proposed in the present work shall be expanded in scope to the solution of system of nonlinear fractional order differential and integral equations.

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