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MODELING MOMENTS OF INSURANCE CLAIM SIZE UNDER DIRAC-DELTA FUNCTION

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Abstract: **In general business insurance, the maximum accumulated amount of losses retained by the insured under deductible policy modifications is usually set as part of the terms and conditions of the policy documents. However, the risk retention of the insured under deductible is limited for every loss event and on a yearly arrangement. A policy without deductible coverage modifications may encounter excessive large claim amount. The insurer only takes over payment of claims given that the aggregate limit is exceeded. This paper develops an analytical framework for evaluating the effect of structural properties of dirac-delta on insurance risk variables with deductible clauses. The aim is to derive models for the moments and variance of insurance claim size in a loss event. In order to achieve this and create analytically sound and useful theoretical basis of investigating actuarial risk functions, the general properties of dirac-delta is first examined in respect of probability density function after some operations. We then obtained insurance claim severity and variance models for an arbitrary policy in general insurance business under deductible coverage modifications which is meant to disapprove nuisance claims and control problem of moral hazard. The results have been clearly stated and proved as part of our findings which may have significant implications on policy underwriting efficiency and decisions.**

***Keywords**:* indemnity; risk; premium; deductible; severity; maximum loss

1. INTRODUCTION TO SINGULARITY FUNCTIONS

Following [1], the dirac-delta function plays active roles in many areas of mathematical disciplines under the appropriate limit especially in the evaluation of asymptotic integrals involved in quantum mechanics. In view of [2], an unparalleled application of dirac-delta was demonstrated in mathematical statistics and probability theory under univariate and multivariate framework. The unit impulse function otherwise called dirac-delta $\delta(x)$ as described in [1,7,12] and which finds applications recently in actuarial statistics is a distribution function rather than a true function which is only defined within an integral on the extended real line. Following [1], it is observed that $\delta(x)$ has singularity at a point on the real line where the integral over the extended real line of the product of a function and dirac-delta produces the functional value of the function at that point. Following the definitions in [1,2,3], dirac-delta permits wider spectrum of application to describe the singularity characteristics of probability distributions used in statistical mechanics especially in quantum theory. In [2,3-4,8-9], we observe different techniques of using dirac-delta function to deal with the differential co-efficient of discontinuous functions. Dirac-delta function is a generalized function such that for a nice function $f(.)$ whose derivatives $f'(x), f''(x),...$ all tends to zero as $x \to \infty$, $\int_{\Re} f(x)\delta(x)\,dx = f(0)$, it then follows that $\int_{\Re} f(x)\delta(x-c)\,dx = \int_{\Re} f(x+c)\delta(x)\,dx = f(c)$. Divergent integral s of the type $\int e^{isx} ds = 2\pi\delta(x), i^2 = -1$ is an example of generalized function. From the paragraph above

$$\int dx f(x) \int e^{isx} ds = \int \sqrt{2\pi}\, ds \int \frac{1}{\sqrt{2\pi}} f(x) e^{isx} ds$$

$FT(f(x))$ is $f^{\circ}(\gamma) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\gamma} dx$

and $IVFT(f(x))$

is $f^{\bullet}(x) = \frac{1}{\sqrt{2\pi}} \int f(\gamma) e^{ix\gamma} d\gamma$

$\int dx f(x) \int e^{isx} ds = \int \sqrt{2\pi} f^{\bullet}(s) \times 1 ds$ since $\frac{2\pi}{\sqrt{2\pi}} = \sqrt{2\pi}$ and $e^0 = 1$, then

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$$\int dxf(x)\int e^{isx}ds = \int \frac{2\pi}{\sqrt{2\pi}} f^{\bullet}(s)e^{-is\times 0}ds$$

$$= 2\pi f(0)$$

The dirac-delta function is defined in [1-6, 8, 12] as follows

$$\delta_\eta(x-\omega) = \delta(x-\omega) = \begin{cases} \infty, \text{if}, x = \omega \\ 0, \text{if}, x \neq \omega \end{cases} \text{ and}$$

$\int_{-\infty}^{\infty} \delta(x-\omega)dx = \int_{0}^{\infty} \delta(x-\omega)dx = 1$, the integral is defined over the extended real line. If $\omega = 0$, we have

$$\delta(x-0) = \begin{cases} \infty, \text{if}, x = 0 \\ 0, \text{if}, x \neq 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(x-0)dx = 1$$

$\delta(x-0)f(x) = \delta(x-0)f(0)$, furthermore

$$\delta(x-\omega)f(x) = \delta(x-\omega)f(\omega)$$

2. MATERIAL AND METHODS

In this section, we use second order differential equation to explain how direct-delta function evolves, as most problems in actuarial risk literature encounter derivation of models for general insurance and casualty. It is on this basis that we use singularity functions to investigate the behavior of actuarial density functions to enable us obtain models applicable in general insurance business.

The linear equation, $a_1y'' + a_2y' + a_3y = f(s)$, is deeply rooted in many subfield of actuarial discipline, especially in financial engineering where it has been used to analyze the term structure and varying time parameters of interest rates by setting the forcing function $f(s) = 0$ and further assuming that the homogeneous differential equation $a_1y'' + a_2y' + a_3y = 0$ has equal real roots with constant co-efficient $a_i, i = 1,2,3$. Following [3,5,8,11-12], one of the most simple but striking application of integral transform occurs in the treatment of linear differential equations with jump discontinuities or discontinuous forcing functions especially in the analysis of circuit problems and mechanical vibrations.

In the second order differential equation above, $f(s)$ is a measure of forcing term and the total area under the curve, $\int_{-\infty}^{\infty} f(s)ds = \lim_{a\to\infty}\int_{-a}^{a} f(s)ds$ is the impulsive force.

(1)

We define the function

$$\delta_\eta(s-s_0) = \begin{cases} \frac{1}{2\eta}, \text{if}, s_0 - \eta < s < s_0 + \eta \\ 0, \text{if, otherwise} \end{cases}$$

and $\int_{-\infty}^{\infty} \delta_\eta(s-s_0)ds = 1$ (2)

where η is a small positive number,

$$\int_{s_0-\eta}^{s_0+\eta} \delta_\eta(s-s_0)f(s)ds = \int_{s_0-\eta}^{s_0+\eta} \frac{1}{2\eta}f(s)ds =$$

$$(s_0 + \eta - s_0 + \eta)\frac{1}{2\eta}f(\bar{s})$$

(3)

$\int_{s_0-\eta}^{s_0+\eta} \frac{1}{2\eta}f(s)ds = \frac{2\eta}{2\eta}f(\bar{s}) = f(\bar{s})$, using the mean value theorem (4)

$$\eta \to 0, \delta_\eta(s-s_0) \to \delta(s-s_0), \bar{s} \to s_0$$

$$\lim_{\eta\to 0}\left[\int_{-\infty}^{\infty} \delta_\eta(s-s_0)f(s)ds\right] = \lim_{\eta\to 0} f(\bar{s}) =$$

$$\int_{-\infty}^{\infty} \lim_{x\to\infty}\left[\delta_\eta(s-s_0)f(s)\right]ds \quad (5)$$

$\int_{-\infty}^{\infty} \delta(s-s_0)f(s)ds = f(s_0)$ and setting $s_0 = 0$, then (6)

$$\int_{-\infty}^{\infty} \delta(s-0)f(s)ds = f(0)$$

(7)

The integral value 1 and the limiting value 0 both define the value of dirac-delta function δ which has a value 1 when s = 0 and 0 if otherwise.

if, $\int_{-\infty}^{\infty} \delta(s-s_0)f(s)ds \approx f(\bar{s})$, $\delta(s-s_0)$ is the kernel of the integral transform describing the dimensions of a rectangular paralleliped of length

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2η and height $\frac{1}{2\eta}$ and centered at s_0 so that the area of the paralleliped will be 1. $\delta(s-s_0)$ isolates the real value of $f(s)$ at some prescribed point s_0 by the normalizing property of dirac-delta function, $\delta(s)=\delta(-s)$ and $\delta(s-\bar{s})=\delta(\bar{s}-s)$ (8)

$$\int_{-\infty}^{\infty}\delta(s-\bar{s})ds=\int_{-\infty}^{\infty}\delta(s-\bar{s})ds=1\Rightarrow$$

$$\int_{-\infty}^{\infty}\delta(t)dt=1,\ \text{when}\ t=(s-\bar{s}) \quad (9)$$

2.1 Application Of Dirac-delta To Probability Density Functions

The function $f(x)$ defines the final pay-off to a unit linked insurance which is maturing at time x. Consider

$x_1 < s_0 < x_2$,

$$\int_{x_1}^{x_2}\delta(x-s_0)f(x)dx=\int_{x_1}^{x_2}\delta(x-s_0)f(s_0)dx$$

$$=f(s_0)\int_{x_1}^{x_2}\delta(x-s_0)dx \quad (10)$$

$$\int_{x_1}^{x_2}\delta(x-s_0)f(x)dx=f(s_0)\times 1$$

$$=\partial(\delta(x))=f(s_0) \quad (11)$$

$\partial(.)$ is the Laplace transform of $\delta(x)$

Again, substituting $s_0=0$, we have

$$\int_{x_1}^{x_2}\delta(x-0)f(x)dx=f(0)\times 1=f(0)\times 1$$

$$=f(0) \quad (12)$$

If $H(x)$ is the unit step function, then

$$\delta(x-k)=\frac{dH(x-k)}{dx} \quad (13)$$

Now $F_X(x)$ is the distribution function of a random risk X with the property that $\frac{dF_X(x)}{dx}=f_X(x)$ (14)

Define $F_X(x)=\sum_{x_i\in\Omega_X}P(x_i)H(x-x_i)$, where Ω_X is the support of X. By the above property,

$$\frac{dF_X(x)}{dx}=\frac{d}{dx}\left[\sum_{x_i\in\Omega_X}P(x_i)H(x-x_i)\right]=,$$

$$\left[\sum_{x_i\in\Omega_X}P(x_i)\frac{d}{dx}\left[H(x-x_i)\right]\right] \quad (15)$$

so that the probability density function is obtained as

$$\frac{dF_X(x)}{dx}=f_X(x)=\sum_{x_i\in\Omega_X}P(x_i)\delta(x-x_i) \quad (16)$$

where $\Omega_X=\{x_i\}\ i=1,2,3,\ldots$ and $P(x_i)$ are the mass points. Let us now compute the moments of the random risk X.

$$E(X)=\int_{-\infty}^{\infty}xf_X(x)dx=\int_{-\infty}^{\infty}\left[\sum_{x_i\in\Omega_X}P(x_i)\delta(x-x_i)\right]xdx \quad (17)$$

$$E(X)=\int_{-\infty}^{\infty}\left[\sum_{x_i\in\Omega_X}P(x_i)\delta(x-x_i)\right]xdx=$$

$$\left[\sum_{x_i\in\Omega_X}P(x_i)\int_{-\infty}^{\infty}x\delta(x-x_i)dx\right]$$

since $f_X(x)=x$ (18)

$$E(X)=\int_{-\infty}^{\infty}\left[\sum_{x_i\in\Omega_X}P(x_i)\delta(x-x_i)\right]xdx=\left[\sum_{x_i\in\Omega_X}x_iP(x_i)\right] \quad (19)$$

$$E(X^2)=\int_{-\infty}^{\infty}x^2f_X(x)dx=\int_{-\infty}^{\infty}\left[\sum_{x_i\in\Omega_X}P(x_i)\delta(x-x_i)\right]x^2dx \quad (20)$$

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$$E\left(X^2\right)=\left[\sum_{x_i\in\Omega_X} P\left(x_i\right)\int_{-\infty}^{\infty} x^2\delta\left(x-x_i\right)dx\right]$$
$$=\sum_{x_i\in\Omega_X} x_i^{\,2}P\left(x_i\right) \tag{21}$$

$$Var\left(X\right)=E\left(X^2\right)-E\left(X\right)E\left(X\right)=$$
$$\left[\sum_{x_i\in\Omega_X} x_i^{\,2}P\left(x_i\right)\right]-\left[\sum_{x_i\in\Omega_X} x_iP\left(x_i\right)\right]^2 \tag{22}$$

hence the first two moments and variance are well defined

3. ACTUARIAL STATISTICS USING DIRAC-DELTA KERNEL

The goal of this section is to obtain the conditional density of a random risk in terms of dirac-delta

In view of [13-15], we Let $X_j, j=1,2,3,...m$ be the size of random loss with frequency $f_{X_j}(x), j=1,2,3,...m$

and such that if $\sum_{j=1}^{m}P_j=1$. We let

$$f_{X_1}\left(x_1\right)=P_1\delta\left(X_1-x_1^{\,*}\right)+P_2\delta\left(X_1-x_2^{\,*}\right)+...+$$
$$P_m\delta\left(X_1-x_m^{\,*}\right)=\sum_{j=1}^{m}P_j\delta\left(X_1-x_j^{\,*}\right) \tag{23}$$

where $x_j^{\,*}$ are the functional values of X_1

hence, $f_{X_1,X_2}\left(x_1,x_2\right)=$

$$\sum_{j=1}^{n}\sum_{k=1}^{m}P_{jk}\delta\left(x_1-x_j^{\,*}\right)\delta\left(x_2-z_k^{\,*}\right) \tag{24}$$

where $z_k^{\,*}$ are the functional values of random risk x_2

$$f_X\left(x\right)=f_{X_1,X_2}\left(x_1,x_2\right)=\frac{\partial^2F_{X_1,X_2}\left(x_1,x_2\right)}{\partial x_1\partial x_2}=$$
$$\sum_{j=1}^{n}\sum_{k=1}^{m}P_{jk}\delta\left(X_1-x_j^{\,*}\right)\delta\left(X_2-x_k^{\,*}\right) \tag{25}$$

$$f_1\left(x_1\right)=\sum_{x_2\in\Omega_2} f_{X_{11}X_2}\left(x_1,x_2\right)$$
$$f_1\left(x_1\right)=\sum_{j=1}^{n}\sum_{k=1}^{m}P_{jk}\delta\left(X_1-x_j^{\,*}\right)\delta\left(X_2-x_k^{\,*}\right) \tag{26}$$

$$f_2\left(x_2\right)=\sum_{x_1\in\Omega_1} f_{X_{11}X_2}\left(x_1,x_2\right)$$
$$f_2\left(x_2\right)=\sum_{k=1}^{n}\sum_{j=1}^{m}P_{kj}\delta\left(X_1-x_j^{\,*}\right)\delta\left(X_2-x_k^{\,*}\right) \tag{27}$$

$$\sum_{x_1\in\Omega_1} f_1\left(x_1\right)=\sum_{x_1\in\Omega_1}\sum_{x_2\in\Omega_2} f_{X_{11}X_2}\left(x_1,x_2\right) \tag{28}$$

$$\sum_{x_2\in\Omega_2} f_2\left(x_2\right)=\sum_{x_2\in\Omega_2}\sum_{x_1\in\Omega_1} f_{X_{11}X_2}\left(x_1,x_2\right) \tag{29}$$

$$f_{X_2|X_1}\left(x_2|x_1\right)=\frac{f_{X_1,X_2}\left(x_1,x_2\right)}{f_1\left(x_1\right)} \tag{30}$$

This is a function of x_2 but x_1 is arbitrarily fixed value of $f_1\left(x_1\right)>0$

$$\sum_{x_2} f_{X_2|X_1}\left(x_2|x_1\right)=\frac{\sum_{x_2\in\Omega_2} f_{X_{11}X_2}\left(x_1,x_2\right)}{f_1\left(x_1\right)} \tag{31}$$

$$\sum_{x_1} f_{X_1|X_2}\left(x_1|x_2\right)=\frac{\sum_{x_1\in\Omega_1} f_{X_{11}X_2}\left(x_1,x_2\right)}{f_2\left(x_2\right)} \tag{32}$$

$$f_{X_1|X_2}\left(x_1|x_2\right)=\frac{f_{X_1,X_2}\left(x_1,x_2\right)}{f_2\left(x_2\right)} \tag{33}$$

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$$f_{X_2|X_1}\left(x_2|x_1\right)=\frac{f_{X_1,X_2}\left(x_1,x_2\right)}{f_1\left(x_1\right)}=$$

$$\frac{\frac{\partial^2 F_{X_1,X_2}\left(x_1,x_2\right)}{\partial x_1 \partial x_2}}{f_1\left(x_1\right)}=\frac{\Pr\left(X_1=x_1,X_2=x_2\right)}{\Pr\left(X_1=x_1\right)} \quad (34)$$

$$f_{X_2|X_1}\left(x_2|x_1\right)=\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}P_{jk}\delta\left(X_1-x_j^*\right)\delta\left(X_2-x_k^*\right)}{\sum_{j=1}^{m}P_j\delta\left(x_1-x_j^*\right)}. \quad (35)$$

$$f_1\left(x_1\right)=\int_{-\infty}^{\infty}f_{X_1 1 X_2}\left(x_1,x_2\right)dx_2,\ f_2\left(x_2\right)=\int_{-\infty}^{\infty}f_{X_1 1 X_2}\left(x_1,x_2\right)dx_1 \quad (36)$$

$$\int_{-\infty}^{\infty}f_1\left(x_1\right)dx_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{X_1 1 X_2}\left(x_1,x_2\right)dx_1dx_2$$

$$f_{X_1|X_2}\left(x_1|x_2\right)=\frac{f_{X_1,X_2}\left(x_1,x_2\right)}{f_2\left(x_2\right)} \quad (37)$$

The joint density function,

$$f_{X_1,X_2}\left(x_1,x_2\right)=\iint f\left(x_1,x_2\right)\delta\left(x-x_1,x-x_2\right)dx_1dx_2 \quad (38)$$

$$f_{X_1|X_2}\left(x_1|x_2\right)=\frac{f_{X_1,X_2}\left(x_1,x_2\right)}{f_2\left(x_2\right)} \quad (39)$$

$$f_{X_1|X_2}\left(x_1|x_2\right)=\frac{\iint f\left(x_1,x_2\right)\delta\left(x_1-x^*,x_2-z^*\right)dx_1dx_2}{\int f_{X_1 1 X_2}\left(x_1,x_2\right)\delta\left(x_2-x^*\right)dx_1} \quad (40)$$

This is the conditional density for the random risk in the continuous sense

4. THE INSURANCE RISK MEASURES USING DIRAC-DELTA KERNEL

In the following two sections, we derive the expected claim and variance. In [14,16-18], it was reported that with a deductible clause, the scheme holder will personally assume the proportion of excess premium payable and claims which occurs during the deductible policy period. However, drawing inference from [19-22] it is possible that a scheme holder may mitigate the policy cost if the insured peril is better than the mean risk in the rating categories of the insurance firm. Deductible clauses are used by insurer where it experiences severe loss profile on an insured to swap a fraction of the loss frequency exposure from the insurance firm to the scheme holder. An insured scheme with a per loss deductible c will repudiate claim payments whenever the claim of size X falls short of or equal to the deductible c. However when the claim value rises above the value c, the underwriter will pay the excess (X - c). The amount of claim size in the loss event is defined by

$$X_L=\begin{cases}0, X\le c\\ X-c, X>c\end{cases} \qquad X_L=\left(X-c\right)_+,\ \text{where}$$

$$X_+=\begin{cases}0, X\le 0\\ X, X>0\end{cases} \quad (41)$$

$$E\left(X_L\right)=E\left(X-c\right)_+=\int_0^{\infty}\Pr\left(X>c+x\right)dx \quad (42)$$

$$\Pr\left(X_L=0\right)=F_X\left(c\right) \quad (43)$$

Following, [13-14], X_L has a probability mass point at zero of $F_X\left(c\right)$

$$f_{X_L}\left(x\right)=f_X\left(x+c\right)\ \text{for } x>0 \quad (44)$$

The expected value function allows us to assess which losses from the risks, the insurance firm will bear in quantitative terms

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$$E(X_L) = \int_{-\infty}^{\infty} (X-c) f_{X_L}(x) dx$$

$$= \int_{0}^{\infty} (X-c) f_{X_L}(x) dx = E(X-c)_+$$

$$E(X_L) = \int_{0}^{\infty} \Pr(X > c+x) f_{X_L}(x) dx$$

$$E(X_L) = \int_{0}^{\infty} (X-c) f_{X_L}(x) dx = E(X-c)_+$$

$$= \int_{0}^{\infty} \Pr(X > c+x) f_{X_L}(x) dx$$

(45)

$$\vartheta = c + x \Rightarrow dx = d\vartheta$$

(46)

We can determine the expected value of loss as,

$E(X_L) = \int_{c}^{\infty} \Pr(X > \vartheta) f_{X_L}(x) d\vartheta$ by the requirement of deductible

It is observed in [14,18, 20-21] that the indemnity function M(L), = $E(X-c)_+ = \max((X-c),0)$ and $\Pr((X-c)_+ > x) = \Pr(X > x+c)$ (47)

$$E(X_L) = \int_{c}^{\infty} (X-c) \sum_{j=1}^{m} P_j \delta(X - x_j^*) dx = \sum_{j=1}^{m} P_j \int_{c}^{\infty} (X-c) \delta(X - x_j^*) dx$$

(48)

$$E(X_L) = \sum_{j=1}^{m} P_j (x_j^* - c) = \sum_{j=1}^{m} P_j x_j^* - c \sum_{j=1}^{m} P_j$$

$E(X_L) = \sum_{j=1}^{m} P_j x_j^* - c \sum_{j=1}^{m} P_j$, recall that $\sum_{j=1}^{m} P_j = 1$

$$E(X_L) = \sum_{j=1}^{m} P_j x_j^* - c$$

(49)

This describes the expected claim liability under the deductible coverage modifications.

Insurance data use knowledge of frequency, which measure the number of times that claims occur in addition to severity, the claim size. Frequency-severity application is an important determinant in actuarial modeling due to the nature of: insurance policies; the profile of data which insurance firms keep and insurance regulatory framework. It is observed in [23,25-27] that at the basic threshold, insurance firms accept premiums with a promise to indemnify the scheme holder over the occurrence of an insured uncertain risk. This indemnity describes the cost of claim, the severity of which defines a major benefit outgo to an insurance firm and is a palliative reinstatement measure to reimburse the scheme holder. Frequency-severity models define the actuarial method of obtaining the expected number of claims which an insurance firm may likely experience in a given period and the cost of average claim. In frequency-severity technique, past data profile is used to model the estimated average number of claims and the average cost per claim. A high frequency is a signal that the insurance firm expects a large number of claims. Furthermore, in view of [17,19,24,27] thus, a large severity claim will be more expensive than an average claim but a small severity claim will be less expensive than the average claim. Actuaries apply advanced mathematical models to obtain the probability that insurance firm will pay out a claim and summarize insurance data set which will be subsequently needed and properly interpreted for underwriting decision process. Obtaining actuarial model to emphasize rate differentials may not be apparent since policy holder behavior may influence the frequency for instance the number of accidents and the severity of the accident.

Now, we define the second moment as follows

$$E(X^2{}_L) = \int_{-\infty}^{\infty} (X-c)^2 f_{X_L}(x) dx$$

(50)

Since density is only defined on the real line, we integrate from zero to infinity

$$E(X^2{}_L) = \int_{0}^{\infty} (X-c)^2 f_{X_L}(x) dx$$

(51)

But by the definition of deductible, we integrate from c to infinity

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$$E\left(X^2{}_L\right)=\int_c^{\infty}(X-c)^2\sum_{j=1}^{m}P_j\delta\left(X-x_j{}^*\right)dx$$

(52)

$$E\left(X^2{}_L\right)=\sum_{j=1}^{m}P_j\int_c^{\infty}(X-c)^2\delta\left(X-x_j{}^*\right)dx$$

$$E\left(X^2{}_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*-c\right)^2=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2$$

$$-2c\sum_{j=1}^{m}P_jx_j{}^*+\sum_{j=1}^{m}P_jc^2$$

(53)

$$E\left(X^2{}_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2-2c\sum_{j=1}^{m}P_jx_j{}^*+c^2$$

(54)

We now compute the variance of the random claim size under the deductible policy contract which accounts for the fluctuations of indicators and appraise the degree of variations of outcome produced from the model. The variance of X will likely fall within many standard deviations of E(X) so that small variance will lead to prectitable probability outcome especially when computing the probability that an insurance firm will make aggregate loss or profit over all its insured schemes.

$$\mathrm{Var}\left(X_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2-2c\sum_{j=1}^{m}P_jx_j{}^*+$$

(55)

$$\sum_{j=1}^{m}P_jc^2-\left(\sum_{j=1}^{m}P_jx_j{}^*-\sum_{j=1}^{m}cP_j\right)^2$$

$$\mathrm{Var}\left(X_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2-2c\sum_{j=1}^{m}P_jx_j{}^*+\sum_{j=1}^{m}P_jc^2-$$

$$\left(\sum_{j=1}^{m}P_jx_j{}^*\right)^2+2\sum_{j=1}^{m}P_jx_j{}^*\sum_{j=1}^{m}cP_j-\left(\sum_{j=1}^{m}cP_j\right)^2$$

$$\mathrm{Var}\left(X_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2-\left(\sum_{j=1}^{m}P_jx_j{}^*\right)^2$$

$$+2\sum_{j=1}^{m}P_jx_j{}^*\sum_{j=1}^{m}cP_j-2c\sum_{j=1}^{m}P_jx_j{}^*+\sum_{j=1}^{m}P_jc^2$$

$$-\left(\sum_{j=1}^{m}cP_j\right)^2$$

$$\mathrm{Var}\left(X_L\right)=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2-\left(\sum_{j=1}^{m}P_jx_j{}^*\right)^2+2c\sum_{j=1}^{m}P_jx_j{}^*$$

$$-2c\sum_{j=1}^{m}P_jx_j{}^*+c^2-c^2=\sum_{j=1}^{m}P_j\left(x_j{}^*\right)^2-\left(\sum_{j=1}^{m}P_jx_j{}^*\right)^2$$

(56)

5. THE RATE OF CHANGE OF PREMIUM TO THE DEDUCTIBLE

The expected indemnity is the ratio of the premium due to a constant

$\Phi=\left(1+\phi\right)$ and hence

$\dfrac{\eta}{\left(1+\phi\right)}=E\left(M\left(L\right)\right)$ where $\eta\left(c\right)$ is the premium chargeable and a differentiable function of c and $\left(1+\phi\right)$ is the loading factor, and $M\left(L\right)$ is the maximum loss $0<M\left(L\right)<x^{**}$

$\eta=\Phi E\left(M\left(L\right)\right)$ implies

$$\eta=\Phi\int_c^{x^{**}}\left(x-c\right)dF_{X_L}\left(x\right)=\Phi\int_c^{x^{**}}\left(x-c\right)f_{X_L}\left(x\right)dx$$

$x^{**}<n_0$

(57)

$$\eta=\Phi\int_c^{x^{**}}\left(x-c\right)dF_{X_L}\left(x\right)=\Phi\int_c^{x^{**}}\left(x-c\right)\sum_{j=1}^{m}P_j\delta\left(X-x_j{}^*\right)dx$$

$$\eta=\Phi P_j\sum_{j=1}^{m}\int_c^{x^{**}}\delta\left(X-x_j{}^*\right)\left(x-c\right)$$

(58)

differentiating the right hand side of (57) with respect to c noting the lower limit is a function of c, we have

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$\frac{\partial \eta}{\partial c} = -\Phi\left(1 - f_{X_L}(c)\right)$ so for the insurance agent's preference of deductible c, we can impose that

$$\left(1 - f_{X_L}(c)\right) < \frac{1}{\Phi} \text{ implies } \frac{\partial \eta}{\partial c} + 1 > 0 \tag{59}$$

$$f_{X_L}(c) > 1 - \frac{1}{\Phi}$$

and thus the additional cost of reducing the deductible by one unit of money cannot be more than the increase in final wealth when $x > c$

Now if $f_{X_L}(x)$ is the claim frequency for claims of size x, then $\int_{-\infty}^{\infty} f_{X_L}(x)\,dx = 1$ and the expected claim size C is

$$E(c) = \int_{-\infty}^{\infty} x f_{X_L}(x)\,dx .$$

Therefore, the proportion of ground up losses in excess of deductible c, can be expressed as

$$\frac{E(c) - c \times 1}{E(c)} = \frac{E(c) - c\int_{-\infty}^{\infty} f_{X_L}\,dx}{E(c)} = \frac{\int_{-\infty}^{\infty} (x-c) f_{X_L}\,dx}{\int_{-\infty}^{\infty} x f_{X_L}\,dx} \tag{60}$$

$$= \frac{\int_{-\infty}^{\infty} (x-c)\sum_{j=1}^{m} P_j \delta\left(X - x_j^*\right)dx}{\int_{-\infty}^{\infty} x\sum_{j=1}^{m} P_j \delta\left(X - x_j^*\right)dx} = \frac{\sum_{j=1}^{m}\int_{0}^{\infty} (x-c) P_j \delta\left(X - x_j^*\right)dx}{\sum_{j=1}^{m}\int_{0}^{\infty} x P_j \delta\left(X - x_j^*\right)dx} = \frac{\sum_{j=1}^{m}\left(x_j^* - c\right)P_j}{\sum_{j=1}^{m} x_j^* P_j} \tag{61}$$

$$\frac{\sum_{j=1}^{m}\int_{0}^{\infty} (x-c) P_j \delta\left(X - x_j^*\right)dx}{\sum_{j=1}^{m}\int_{0}^{\infty} x P_j \delta\left(X - x_j^*\right)dx} = \frac{\sum_{j=1}^{m}\left(x_j^* - c\right)P_j}{\sum_{j=1}^{m} x_j^* P_j} = 1 - \frac{\sum_{j=1}^{m} c P_j}{\sum_{j=1}^{m} x_j^* P_j} = 1 - \frac{c}{\sum_{j=1}^{m} x_j^* P_j} \tag{62}$$

6. CONCLUSION

The dirac-delta function has been successfully applied in this paper. The goal is to draw attention to several notable applications of this function in actuarial statistics. Some of these applications include a unified density representation of the distribution of a function of one or two random risks, severity function which may be discrete or continuous in terms of its non-central moments. In this paper, we have presented different properties of dirac-delta function as applicable in actuarial problems, provided with simple proofs. Part of the motivation for using dirac-delta functions, is its ability to permit alternative technique to obtain analytically useful models for actuarial risk function. In this paper, we have applied the dirac-delta function to obtain:

- The expected claim severity under deductible policy of general insurance using first moment as reported in (49)
- The second moment under deductible policy contract of general insurance as reported in (54)
- The variance of the loss event under the deductible coverage modifications obtained and reported in equation (56)
- The first two moments and variance of an arbitrary random risk is reported in (19), (21) and (22).

The results obtained so far on insurance claims size has significant application in insurance practice thereby adding value to actuarial literature. The correct estimation of frequency and severity of insurance claims allows an insurance firm to meet payments of claims as they occur and to meet daily operational costs. This paper has demonstrated interesting analytical model of deep concern to actuarial literature which was accomplished under probability assumptions. In this paper, we have also shown how dirac-delta could be applied to describe

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probability density of a random risk leading to conditional density. Thus, in explaining a common basis of applying specialized functions to study the behavior of actuarial functions, the dirac-delta technique has been applied to formulate actuarial model regarding claim severities and the variance function. Worthy of note is that this technique offers an excellent framework in investigating the behavior of both discrete and continuous actuarial functions. Future study can be geared towards an empirical result where there is availability of insurance data for the analysis of model parameters

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