Combinatorial Properties on Subsemigroup of Order-decreasing Alternating Semigroup Using Three Distinct Combinatorial Functions; New Approach to Generalization

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Abstract: Let ζ_η be the charts on $\Omega_\eta = \{1,2,\cdots,\eta\}, \Lambda_\eta^\zeta$ be the alternating semigroups, $\mathcal{D}\Lambda_\eta^\zeta$ be the subsemigroup of order decreasing alternating semigroup and $\wp(S)$ be the set of nilpotents element of a semigroup $\mathcal{D}\Lambda_\eta^\zeta$ on Ω_η.

Let $\xi(\eta;\sigma,\Im) = |\{\psi \in \zeta: \ell(\psi) = \sigma \wedge \tau(\psi) = \Im\}|, \xi(\eta;\Im) = |\{\psi \in \zeta: \tau(\psi) = \Im\}|$ and $\xi(\eta;\sigma) = |\{\psi \in \zeta: \ell(\psi) = \sigma\}|$.

Analysing problems of combinatorial nature arise naturally in the study of transformation semigroup. Combinatorial properties of many transformation semigroups and its subsemi-groups have been studied with interesting and delightful results obtained.

This paper investigated combinatorial properties on subsemigroup of order-decreasing alternating semigroups in which combinatorial functions $\xi(\eta;\sigma)$, $\xi(\eta;\Im)$ and $\xi(\eta,\sigma,\Im)$ were used to derive some triangular arrays of numbers and some combinatorial results for each function were established. More so, the results on $|\mathcal{D}\Lambda_\eta^\zeta|$ and $|\wp(\mathcal{D}\Lambda_\eta^\zeta)|$ were also obtained and generalized.

Keywords: Even charts, nilpotent, fix, height, triangular array

1. INTRODUCTION AND PRELIMINARIES

Let ζ_η be the charts on $\Omega_\eta = \{1,2,\cdots,\eta\}, \Lambda_\eta^\zeta$ be the alternating semigroups, $\mathcal{D}\Lambda_\eta^\zeta$ be the subsemigroup of order decreasing alternating semigroup and $\wp(S)$ be the set of nilpotents element of a semigroup $\mathcal{D}\Lambda_\eta^\zeta$ on Ω_η. Let $\xi(\eta;\sigma,\Im) = |\{\psi \in \zeta: \ell(\psi) = \sigma \wedge \tau(\psi) = \Im\}|, \xi(\eta;\Im) = |\{\psi \in \zeta: \tau(\psi) = \Im\}|$ and $\xi(\eta;\sigma) = |\{\psi \in \zeta: \ell(\psi) = \sigma\}|$.

The map $\psi: Dom\psi \subseteq \Omega_\eta \rightarrow Im\psi \subseteq \Omega_\eta$ is said to be full or total if Dom $\psi = \Omega_\eta$, partial if Dom $\psi \subseteq \Omega_\eta$ or else it is called strictly partial. The set of all partial transformation on n-object form a semigroup under the usual composition of transformation. Let $\mathcal{T}_\eta, \mathcal{P}_\eta$ and ζ_η be the full or total, partial and partial one-to-one on Ω_η respectively. "Reference [25] called the semigroups ζ_η charts. These are the three essential parts of transformation semigroups which were introduced in [19].

A transformation ψ in ζ_η is called ("alternating semigroup") if it can be expressed as a product of an even number of transpositional or a product of any number of circuits/paths of odd length. A transformation ψ in a semigroup $\mathcal{D}\Lambda_\eta^\zeta$ is *nilpotent* if there exists $\eta \geq 0$ such that $\psi^\eta = \emptyset$. The idea of an even permutation has been generalized via path notation to one-to-one partial transformation. "Reference [24] explained that each subpermutation $\psi\ on \Omega_\eta$ can be pictured as a digraph on η vertices with p, q and edge of ψ if $p\psi = q$. Each component of such a digraph is called an $orbit$.

The combinatorial properties on different classes of transformation semigroups have been studied by different researchers and many interesting and delightful results have emerged. For instance in [23] looked into the combinatorial properties of the symmetric inverse semigroups , in [8] got some results on combinatorial properties of the alternating groups (Λ_η), in [17] studied some algebraic and combinatorial properties of semigroup of injective partial contraction mappings and isometrics of a finite chain,in [1] studied identity difference transformation semigroups , [2] also studied some semigroups of full contraction mapping on a finite chain. Recently , inspired by the work in [25] , [3-6] obtained some results on combinatorial properties in the semigroup Λ_η^ζ and its subsemigroup $\mathcal{O}\Lambda_\eta^\zeta$ respectively. But the combinatorial properties on subsemigroup $\mathcal{D}\Lambda_\eta^\zeta$ have not be study. This paper is therefore focused on combinatorial properties of subsemigroup $\mathcal{D}\Lambda_\eta^\zeta$.

Analysing problems of combinatorial nature arise naturally in the study of transformation semigroups. The sequences and triangle of numbers regarded as combinatorial gems like the Stirling numbers used by [19], the Factorial used in [26, 28], the Binomial used in [16, 20], the Fibonacci number used in [18], Catalan numbers used in [14] and Lah numbers used in [21, 22], etc., have all characterized in these analysis problems. These problems lead to many numbers in [27] but there are also others that are not yet in it and this underscores the need for the present study.

The theory of semigroups has its scope widened to embrace some other aspects of theoretical computer sciences. This research work and its findings are expected to be beneficial in the areas of automata theory, coding theory, computational theory and formal languages as well as application in the sciences. It can also assist in sorting data and designing better networks. For standard concepts and terms in semigroup, symmetric inverse semigroup,alternating group and semigroups in [19], [15] and [25].

Definition 1.1:[Umar[29]]

$$\mathcal{D}\Lambda_\eta^\zeta = \{\psi \in \mathcal{D}\Lambda_\eta^\zeta : (\forall\, \nu \in Dom\psi)\ \nu\psi \leq \nu\},$$

is the subsemigroup of Λ_η^ζ containing all *order-decreasing alternating semigroup* on Ω_η.

Definition 1.2:

Let $\psi \in \mathcal{D}\Lambda_\eta^\zeta$. Then, the height of ψ is $\ell(\psi) = |Im\psi|$.

Definition 1.3:

Let $\psi \in \mathcal{D}\Lambda_\eta^\zeta$. Then, the fix of ψ is $\tau(\psi) = |F(\psi)|$.
where $F(\psi) = \{\mu \in Dom\psi : \mu\psi = \mu\}$.

Theorem 1.4:[Borwein[9]]

Let $\mathcal{S} = \zeta\mathcal{D}_\eta$. Then $\xi(\eta;\sigma) = S(\eta+1;\eta-\sigma+1)$. where $S(\eta;\sigma)$ is the Stirling number of the second kind.

Proposition 1.5: [Umar,[29]]

Let $\mathcal{S} = \zeta\mathcal{D}_\eta$. Then,

$$\xi(\eta;\Im) = \binom{\eta}{\Im} B_{\eta-\Im}$$

Proposition 1.6:[Borwein [9]]

Let $\mathcal{S} = \zeta\mathcal{D}_\eta$.Then $|\zeta\mathcal{D}_\eta| = B_{\eta+1}$, where $B_{\eta+1}$ is the Bell's number.

Proposition 1.7:[Umar,[28]]

Let $\wp(\zeta\mathcal{D}_\eta) = \{\psi \in (\zeta\mathcal{D}_\eta) : F(\psi) = \emptyset\}$.
Then $|\wp(\zeta\mathcal{D}_\eta)| = B_\eta$.

METHODOLOGY

The following procedures were used in carrying out this research work:

(i) the elements of $\mathcal{D}\Lambda_\eta^\zeta$ were constructed for $1 \leq \eta \leq 6$;

(ii) the combinatorial functions $\xi(\eta;\sigma)$, $\xi(\eta;\Im)$ and $\xi(\eta,\sigma,\Im)$ were used in $\mathcal{D}\Lambda_\eta^\zeta$ to derive some of the triangular arrays of numbers; and
(iii) the pattern of the triangular arrays of numbers obtained were studied and some results were obtained on their combinatorial properties.

2. MAIN RESULTS

Theorem 2.1:

Let $\mathcal{S} = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$\xi(\eta;\sigma_{\eta-1}) = \begin{cases} \binom{\eta}{\eta-1} + \sum_{z,odd}^{\eta-3} \binom{\eta}{z}, & \text{if } \eta \text{ is even} \\ \binom{\eta}{\eta-1} + \sum_{z,even}^{\eta-3} \binom{\eta}{z}, & \text{if } \eta \text{ is odd} \end{cases}$$

Proof. Let z be the fix of ψ i.e $\tau(\psi) = z$. Then the unique path consists of $\eta - z$ points. For the completion ψ^- of ψ to be even we need $\eta - z$ to be odd.

Clearly these are two cases: if η is even then z must be odd and if η is odd then z must be even.

Moreover, the unique path must have at least 3 points $(\eta - z \geq 3)$ implies $z \leq \eta - 3$.

Thus,

$$\xi(\eta;\sigma_{\eta-1}) = 2^{\eta-1} = \begin{cases} \binom{\eta}{\eta-1} + \sum_{z,odd}^{\eta-3} \binom{\eta}{z}, & \text{if } \eta \text{ is even} \\ \binom{\eta}{\eta-1} + \sum_{z,even}^{\eta-3} \binom{\eta}{z}, & \text{if } \eta \text{ is odd} \end{cases}$$

□

Proposition 2.2:

Let $\mathcal{S} = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$\xi(\eta;\sigma) = \begin{cases} 1, & \text{if } \sigma = \eta \\ \binom{\eta}{\eta-1} + \sum_{z,odd}^{\eta-3} \binom{\eta}{z}, & \text{if } \eta \text{ is even and } \sigma = \eta - 1 \\ \binom{\eta}{\eta-1} + \sum_{z,even}^{\eta-3} \binom{\eta}{z}, & \text{if } \eta \text{ is odd and } \sigma = \eta - 1 \\ S(\eta+1;\eta-\sigma+1), & \text{for } 0 \leq \sigma \leq \eta - 2 \end{cases}$$

Proof. It is obvious to see that $\xi(\eta;\sigma_\eta) = \xi(\eta;\eta) = 1, \forall \eta$. If $\sigma = \eta - 1$ and for $0 \leq \sigma \leq \eta - 2$ then the results follow from Theorem 2.1 and Theorem 1.2 respectively for $\sigma < \eta - 2 \subseteq \zeta\mathcal{D}_\eta \in \mathcal{D}\Lambda_\eta^\zeta$.

Lemma 2.3:

Let $\mathcal{S} = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$\xi(\eta;\Im_0) = \xi(\eta;0) = \begin{cases} B_\eta, & \text{if } \eta \text{ is odd} \\ B_\eta - 1, & \text{if } \eta \text{ is even} \end{cases}$$

Proof.

Let

$$\Phi = \begin{pmatrix} \varepsilon_{p+1} & \varepsilon_{p+2} & - & - & -\varepsilon_\eta \\ \zeta_i & \zeta_{i+1} & - & - & -\zeta_{\eta-1} \end{pmatrix}$$

for $F(\Phi) = 0$. It is clear to see that there is a unique path of a nilpotent element at $\ell(\Phi) = \eta - 1 \in \mathcal{D}\Lambda_\eta^\zeta$, then the completion Φ^- of Φ is odd if η is even and even if η is odd. □

Theorem 2.4:

Let $\mathcal{S} = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$\xi(\eta;\Im) = \binom{\eta}{\Im}\xi(\eta - \Im;\Im_0)$$

Proof. Let $\psi \in \mathcal{D}\Lambda_\eta^\zeta$ and let $v_1, v_2, ---, v_\Im \in$ fix of ψ. Since ψ is an order-decreasing even charts, then for $v \in (V_\eta \backslash \{v_1, ---, v_\Im\}) \cap Dom\psi$ we have $v\psi \in V_\eta \backslash \{v_1, ---, v_\Im\}$ and $v\psi < v$. Thus, the restriction of ψ to $V_\eta \backslash \{v_1, ---, v_\Im\}$ posed and is a nilpotent element of $\mathcal{D}\Lambda_\eta^\zeta(V_\eta \backslash \{v_1, ---, v_\Im\})$. The number of nilpotent elements that can be formed by these $\eta - \Im$ elements is $\xi(\eta - \Im;\Im_0)$. Therefore, the result follows in [28], Theorem 4.2].

Theorem 2.5:

Let $\mathcal{S} = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$|\mathcal{D}\Lambda_\eta^\zeta| = B_{\eta+1} - 2^{\eta-1} + 1.$$

Proof. It is well known in [29] that $\sum_{\sigma=0}^{\eta} \xi(\eta;\sigma) = |\mathcal{D}\Lambda_\eta^\zeta|$.

Thus,

$\sum_{\sigma=0}^{\eta} \xi(\eta;\sigma) = \sum_{\sigma=0}^{\eta-2} \xi(\eta;\sigma) + \xi(\eta;\sigma_{\eta-1}) + \xi(\eta;\sigma_{\eta})$ By Proposition 2.2, we have $|\mathcal{D}\Lambda_{\eta}^{\zeta}| = \sum_{\sigma=0}^{\eta-2} S(\eta+1;\eta-\sigma+1) + 2^{\eta-1} + 1$ $|\mathcal{D}\Lambda_{\eta}^{\zeta}| = \sum_{\sigma=0}^{\eta} S(\eta+1;\eta-\sigma+1) - [S(\eta+1,2) + S(\eta+1;1)] + 2^{\eta-1} + 1$ Again, by Theorem 1.2 and Proposition 1.4, we have $|\mathcal{D}\Lambda_{\eta}^{\zeta}| = B_{\eta+1} - [(2^{\eta}-1)+1] + 2^{\eta-1} + 1$ $|\mathcal{D}\Lambda_{\eta}^{\zeta}| = B_{\eta+1} - 2^{\eta-1} + 1$. □

Remark 2.6:

The triangular arrays of numbers $\xi(\eta;\sigma)$ and $\xi(\eta;\Im)$ in $\mathcal{D}\Lambda_{\eta}^{\zeta}$ are not yet listed in [27]. For selected values of these numbers see Table 1 and 2.

TABLE 1. Triangular array of numbers $\xi(\eta;\sigma)$ in $\mathcal{D}\Lambda_{\eta}^{\zeta}$

η/σ	0	1	2	3	4	5	6	$\sum \xi(\eta;\sigma)$
0	1							1
1	1	1						2
2	1	2	1					4
3	1	6	4	1				12
4	1	10	25	8	1			45
5	1	15	65	90	16	1		188
6	1	21	140	350	301	32	1	846

TABLE 2. Triangular array of numbers $\xi(\eta;\Im)$ in $\mathcal{D}\Lambda_{\eta}^{\zeta}$

$\eta/\Im$	0	1	2	3	4	5	6	$\sum \xi(\eta;\Im)$
0	1							1
1	1	1						2
2	1	2	1					4
3	5	3	3	1				12
4	14	20	6	4	1			45
5	52	70	50	10	5	1		188
6	202	312	210	100	15	6	1	846

Lemma 2.7:

Let $\mathcal{S} = \mathcal{D}\Lambda_{\eta}^{\zeta}$. Then,

$\xi(\eta,\sigma_{\eta},\Im_{\eta}) = 1 \forall \eta$

Lemma 2.8:

Let $\mathcal{S} = \mathcal{D}\Lambda_{\eta}^{\zeta}$. Then,

$$\xi(\eta,\sigma_{\eta-1},\Im_0) = \begin{cases} 1, & \text{if } \eta \text{ is odd and } \sigma = \eta - 1 \\ 0, & \text{if } \eta \text{ is even and } \sigma = \eta - 1 \end{cases}$$

Proof. Let $\psi \in \mathcal{D}\Lambda_{\eta}^{\zeta}$ and $\tau(\psi) = \Im$ be such that $\ell(\psi) = \eta - 1$. Then the completion $\psi^{-} of \psi$ is odd if η is even and even if η is odd. By the order-decreasing alternating property, since there is no fixed point, then there is only one element of the $\mathcal{D}\Lambda_{\eta}^{\zeta}$ if η is odd and 0 otherwise. □

Theorem 2.9:

Let $\mathcal{S} = \mathcal{D}\Lambda_{\eta}^{\zeta}$. Then,

$\xi(\eta,\sigma_{\leq \eta-2},0) = S(\eta;\eta-\sigma)$, for $0 \leq \sigma \leq \eta - 2$

Proof. Notice that by virtue of Proposition 1.3, it suffices to establish a bijection between $\wp(\mathcal{D}\Lambda_{\eta}^{\zeta})$ and $\mathcal{D}\Lambda_{\eta-1}^{\zeta}$. So for every $\psi_{\eta} \in \wp(\mathcal{D}\Lambda_{\eta}^{\zeta})$ we associate an $\psi \in \mathcal{D}\Lambda_{\eta-1}^{\zeta}$ by

$\xi(\emptyset) = \emptyset$ and $\xi(\psi_{\eta}) = \psi$

where $z\psi_{\eta} = (z-1)\psi, (z \in dom\psi_{\eta})$

Now since $1 \notin dom\,\psi_{\eta}$ and $\eta \notin dom\,\psi$ then clearly ξ is a bijection, as required. □

Consequently, from Lemma 2.7, Lemma 2.8 and Theorem 2.9 we deduces that,

Corollary 2.10:

Let $\mathcal{S} = \mathcal{D}\Lambda_{\eta}^{\zeta}$. Then,

$$\xi(\eta;\sigma,\Im_0) = \begin{cases} 1, \text{if} \sigma = \eta = \Im \text{and} 0, \text{otherwise} \\ 1, \ \text{if} \eta \text{isoddand} \sigma = \eta - 1 \\ 0, \ \text{if} \eta \text{isevenand} \sigma = \eta - 1 \\ \ S(\eta;\eta-\sigma), \ \text{for} \ \ 0 \le \sigma \le \eta - 2 \end{cases}$$

Proposition 2.11:

Let $S = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$\xi(\eta;\sigma_{\eta-1},\Im) = \begin{cases} \binom{\eta}{\Im}, \ \text{if} \eta \text{isevenand} \sigma = \eta - 1 \\ 0, \ \text{if} \eta \text{isoddand} \sigma = \eta - 1 \end{cases}$$

Proof. Let $\psi \in \mathcal{D}\Lambda_\eta^\zeta$ and $\tau(\psi) = \Im$. Since $\ell(\psi) = \eta - 1$. Then the completion $\psi^{-} of \psi$ is odd if η is even and even if η is odd. Moreover, it follows that there should be no fixed points in $\mathcal{D}\Lambda_\eta^\zeta$. By the order-decreasing alternating property, we see that there are $\binom{\eta}{\Im}$ ways if η is even and 0 if η is odd. □

Theorem 2.12:

Let $S = \mathcal{D}\Lambda_\eta^\zeta$. Then

$$\xi(\eta,\sigma_{\le\eta-2},\Im) = \binom{\eta}{\Im} S(\eta-\Im;\eta-\sigma),$$
$$\text{for} \ \ 0 \le \sigma \le \eta - 2$$

Proof. From Theorem 1.5 we see that for every $\ell(\psi) < \eta - 1$ are all order decreasing symmetric inverse semigroups. Let $\psi \in \mathcal{D}\Lambda_\eta^\zeta$ and $\tau(\psi) = \Im$ be the fixed points of ψ. Then $\Im$ fixed points of ψ can be chosen from Ω_η in $\binom{\eta}{\Im}$ ways. It follows that, on the remaining $\eta - \Im$ points there should be no fixed points in Ω_η. By the order decreasing/symmetric inverse semigroup property we see that there are $\xi(\eta-\Im,\sigma-\Im,\Im_0) = S(\eta-\Im,\eta-\sigma)$ possibilities. □

Consequently, from Theorem 2.9,Proposition 2.11 and Theorem 2.12 we deduces that

Corollary 2.13:

Let $S = \mathcal{D}\Lambda_\eta^\zeta$. Then,

$$\xi(\eta;\sigma,\Im) = \begin{cases} 1, \text{if} \sigma = \eta = \Im \text{and} 0, \text{otherwise} \\ \binom{\eta}{\Im}, \ \text{if} \eta \text{isevenand} \sigma = \eta - 1 \\ 0, \ \text{if} \eta \text{isoddand} \sigma = \eta - 1 \\ \binom{\eta}{\Im} S(\eta-\Im;\eta-\sigma), \ \text{for} \ \ 0 \le \sigma \le \eta - 2 \end{cases}$$

Remark 2.14:

The triangular arrays of numbers $\xi(\eta;\sigma,\Im)$ for $(\Im = 0,1,2)$ in $\mathcal{D}\Lambda_\eta^\zeta$ are not yet listed in [27] which we believe they are new . For some selected values of these numbers see Tables $3,4$ and $5.$

TABLE 3. Triangular array of numbers $\xi(\eta;\sigma,\Im_0)$ in $\mathcal{D}\Lambda_\eta^\zeta$

η/σ	0	1	2	3	4	5	6	$\sum \xi(\eta;\sigma,\Im_0)$
0	1							01
1	1	0						01
2	1	0	0					01
3	1	3	1	0				05
4	1	6	7	0	0			14
5	1	10	25	15	1	0		52
6	1	15	65	90	31	0	0	202

TABLE 4. Triangular array of numbers $\xi(\eta;\sigma,\Im_1)$ in $\mathcal{D}\Lambda_\eta^\zeta$

η/σ	0	1	2	3	4	5	6	$\sum \xi(\eta;\sigma,\Im_1)$
0	0							00
1	0	1						01
2	0	2	0					02
3	0	3	0	0				03
4	0	4	12	4	0			20
5	0	5	30	35	0	0		70
6	0	6	60	150	90	6	0	312

TABLE 5. Triangular array of numbers $\xi(\eta;\sigma,\Im_2)$ in $\mathcal{D}\Lambda_\eta^\zeta$

η/σ	0	1	2	3	4	5	6	$\sum \xi(\eta,\sigma,\Im_2)$
0	0							00
1	0	0						00
2	0	0	1					01
3	0	0	3	0				03
4	0	0	6	0	0			06
5	0	0	10	30	10	0		50
6	0	0	15	90	105	0	0	210

Theorem 2.15 :

Let $\wp(\mathcal{D}\Lambda_\eta^\zeta)$ be the nilpotent element in order-decreasing alternating semigroup on Ω_η. Then

$$|\wp(\mathcal{D}\Lambda_\eta^\zeta)| = \begin{cases} B_\eta, & \text{if}\eta\text{isodd} \\ B_\eta - 1, & \text{if}\eta\text{iseven} \end{cases}$$

Proof. If η is odd the proof follow from Theorem 1.5 and if η is even it follow from Lemma 2.3 respectively. □

Remark 2.16:

The triangular array of number $\xi(\eta;\sigma)$ in $\wp(\mathcal{D}\Lambda_\eta^\zeta)$ are not yet listed in [27]. For selected values of this number see Table 6

TABLE 6. Triangular array of numbers $\xi(\eta;\sigma)$ in $\wp(\mathcal{D}\Lambda_\eta^\zeta)$

η/σ	0	1	2	3	4	5	6	$\sum \xi(\eta;\sigma)$
0	1							01
1	1	0						01
2	1	0	0					01
3	1	3	1	0				05
4	1	6	7	0	0			14
5	1	10	25	15	1	0		52
6	1	15	65	90	31	0	0	202

3. DISCUSSION OF RESULTS

The combinatorial functions of two parameters $\xi(\eta;\sigma)$ and $\xi(\eta;\Im)$ were used to obtain results of Theorem $2.1, Prosition\ 2.2, Lemma\ 2.3$ and Theorem 2.4 respectively. More so Theorem 2.5 generalized the results on $|\mathcal{D}\Lambda_\eta^\zeta|$. Furthermore, the combinatorial function of three parameters $\xi(\eta,\sigma,\Im)$ were also used to obtained results of Lemma $2.7, 2.8$ and 2.9 from which the Theorem 2.10 was deduced. More results were also obtained on $\xi(\eta,\sigma,\Im)$ and stated in Proposition 2.11 and Theorem 2.12 which lead to Corollary 2.13. Lastly, Theorem 2.15 generalized result of nilpotent in $|\mathcal{D}\Lambda_\eta^\zeta|$.

4. CONCLUSION

This paper investigated combinatorial properties on subsemigroup of order-decreasing alternating semigroups in which combinatorial functions $\xi(\eta;\sigma)$, $\xi(\eta;\Im)$ and $\xi(\eta,\sigma,\Im)$ were used to derive some triangular arrays of numbers and some of its combinatorial results for each function were established. Moreso, the results on $|\mathcal{D}\Lambda_\eta^\zeta|$ and $|\wp(\mathcal{D}\Lambda_\eta^\zeta)|$ were also obtained and generalized. It is hereby recommended that the work should be extended to other subsemigroups of alternating semigroups such as; order preserving/order reversing, orientation preserving and other semigroups such as isometrics of injective mapping, identity difference transformation semigroups, etc.

This paper has developed some new triangular arrays which lead to many number of sequences as shown under appendix. The results obtained are not yet listed in [27] which is the largest database of its kind and we believe they are new and can be submitted after the publication of the results listed in this works.

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