FIXED POINT RESULTS ON SEMIGROUP OF ORDER PRESERVING MAPS IN METRIC SPACES

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In this paper, we introduce new classes of subsemigroups of Order Preserving Full Contraction (OCT_η) and Order Preserving Full Contractive (OC^*T_η) mappings respectively in metric spaces. The relationship between the fixed elements of these subsemigroups were thoroughly examined in line with approximate fixed points. We show that, every fixed elements of subsemigroups OCT_η and OC^*T_η has a comparable deterministic fixed points and every subsemigroup OC^*T_η belong to a class of nonexpansive mapping. The existence and uniqueness results of these subsemigroups were also given in a complete metric space (Ξ, ρ) under weakly contractivity conditions which was justified with classical examples.

Keywords: *Semigroup, Full transformation, Order preserving full contraction, Order preserving full contractive, weakly contractive map, right waist, fixed point.*

1. INTRODUCTION ANDPRELIMINARIES

Fixed points of nonlinear operators is an important research branch of nonlinear analysis and has been applied in the study of semigroup theory. The importance of mappings can be judged by the fact that every semigroup is having a structure preserving one-to-one correspondence to a semigroup of mappings. It is well known that the Banach contraction principle is a ground laying result in the fixed point theory which has been used and expanded in many ways.

The theorem of Banach contraction mapping proved in complete metric spaces continues to be a necessity and efficient use in theory as well as applications which guarantees the existence and uniqueness of fixed points of contraction self-maps besides offering a contractive procedure to find the fixed point of the underlying mapping. Several authors have worked on generalized Banach's contraction principle using altering distance functions [5, 10, 16, 19] and also some generalized it using weakly contractivity conditions [9, 21, 22, 26].

A semigroup is simply a pair (S, $*$) and a subset T of a semigroup S is called a subsemigroup of S if it is closed under the binary operation of S.

Let $\chi_\eta = \{1, 2, 3, ..., \eta\}$, then a transformation ϑ: $Dom(\vartheta) \subseteq \chi_\eta \rightarrow Im(\vartheta) \subseteq \chi_\eta$ is called full or total transformation of χ_η if $Dom(\vartheta) = \chi_\eta$. The set of full transformations of χ_η is denoted by T_η. This set T_η of a full transformations of χ_η form a semigroup under composition of mappings called the full transformation semigroup.
The right and left waist of ϑ is denoted and defined by $\omega^+(\vartheta) = \max(Im\vartheta)$ and $\omega^-(\vartheta) = \min(Im\vartheta)$.

More so, the fix of ϑ is denoted and defined by
$$f(\vartheta) = |F(\vartheta)| = |\{\mu \in Dom(\vartheta) : \mu\vartheta = \mu\}| \qquad (1)$$

The semigroup of Order Preserving Full Transformation, denoted by
OT_η is defined as
$OT_\eta = \{\vartheta \in T_\eta : (\forall \mu, \nu \in \chi_\eta) \mu \leq \nu \Rightarrow \mu\vartheta \leq \nu\vartheta\}$
on χ_η.

Let T_η be the full transformation semi group, then OCT_n is called the subsemigroup of Order Preserving Full Contraction and OC^*T_η is called subsemigroup of Order Preserving Full Contractive respectively. A transformation ϑ in OCT_n is said to be a contraction if

$$|\mu\vartheta - \nu\vartheta| \leq |\mu - \nu| \qquad (2)$$

and also ϑ in OC^*T_n is contractive if

$$|\mu\vartheta - \nu\vartheta| \geq |\mu - \nu| \qquad (3)$$

for all $\mu, \nu \in \chi_\eta$.

Most nonlinear problems may be reduced to fixed points of certain operator and the fact that contractive (Lipschitzian) type conditions naturally arise for many of these problems in semigroup theory. Some researchers have obtained fixed point results for certain classes of semigroups [17, 20] and more theorems were also proved on nonexpansive semigroups [1, 2, 8, 18, 23]. Analyzing problems of combinatorial nature arise naturally in the study of transformation semigroup and its subsemigroups in which interesting results have been obtained [3, 12, 14, 24]. Furthermore, results on some classes of semigroups were also established via tropical geometry [4, 25].

This paper aim at unify the fixed elements of the subsemigroups OCT_η and OC^*T_η to the relatively approximate fixed points in metric spaces with the intuition that the deterministic fixed points is assumed to be comparable under weakly contractivity condition. We are concerned with finding conditions on the structure that the elements of these subsemigroups will endow as well as the properties of its corresponding operator $\mathbb{T}$ constructed from χ_η. We make use of geometric properties to rewrite the transformation ϑ on χ_η equivalently as a fixed point problem $x = \mathbb{T}x$ in order to obtain results on the existence and uniqueness of the fixed points on subsemigroups OCT_η and OC^*T_η in (Ξ, ρ). From the existing literature little or no work has been done on these subsemigroups in metric spaces. However, the notion "contractive" in a semigroup connotes nonexpansive in (Ξ, ρ).

For standard concepts and terms in semigroup theory, [13, 15].

The following Definitions, Theorem and Lemma will be useful in our main results.

Definition 1.1 [6]
Let (X, ρ) be a metric space. A map T: $X \to X$ is a contraction if there exist a constant $\kappa \in (0, 1)$ such that $\forall x, y \in X$,
$$\rho\,(Tx, Ty) \le \kappa\,(x, y) \qquad (4)$$

Definition 1.2 [7]
Let (X, ρ) be a metric space. A m ap T: $X \to X$ is called nonexpansive if T is 1−Lipschitzian such that $\forall\, x, y \in X$,
$$\rho\,(Tx, Ty) \le \rho\,(x, y) \qquad (5)$$

Definition 1.3
An Order Preserving Full Contraction Map (OCT_n) is said to be a contraction in a complete metric space (Ξ, ρ) if there exists a corresponding self-map $\mathbb{T}$ in OCT_n such that
(i) $\mathbb{T}$ is closed and
(ii) $\mathbb{T}$ satisfies (4) of Definition 1.1.

Definition 1.4
An Order Preserving Full Contractive Mapping (OC^*T_η) is said to be a nonexpansive map in a complete metric space (Ξ, ρ) if there exists a corresponding self map $\mathbb{T}$ in OC^*T_η such that
(i) $\mathbb{T}$ is closed and
(ii) $\mathbb{T}$ satisfies (5) of Definition 1.2.

Theorem 1.5 [11]
Let (X, ρ) be a complete metric space and T: $X \to X$ be a self-mapping satisfying the inequality
$\varpi(\rho\,(Tx, Ty)) \le \varpi\,(\rho\,(x, y)) - \zeta\,(\rho\,(x, y))$, for all $x, y \in X$ where
ϖ, ζ:$[0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing function with $\varpi\,(\tau\,) = 0 = \zeta(\tau\,)$ if and only if $\tau = 0$. Then T has a unique fixed point.

Lemma 1.6 [5]
Let (M, ρ) be a metric space and $\{y_n\}$ be a sequence in M such that
$\lim_{\eta\to\infty}[\rho\,(y_n, y_{n+1})] = 0$

If $\{y_n\}$ is not a Cauchy sequence in M, then there exist an $\epsilon_O>0$ and sequence of positive integers $m(\kappa)$ and $n(\kappa)$ with $m(\kappa) > n(\kappa) > \kappa$ such that
$\rho(y_{m(k)}, y_{n(k)}) \ge \epsilon_0, \rho(y_{m(k)-1}, y_{n(k)}) < \epsilon_0$ and
(i) $lim_{k\to\infty}\rho(y_{m(k)-1}, y_{n(k)+1}) = \epsilon_0$
(ii) $lim_{k\to\infty}\rho(y_{m(k)}, y_{n(k)}) = \epsilon_0$
(iii) $lim_{k\to\infty}\rho(y_{m(k)-1}, y_{n(k)}) = \epsilon_0$

2. MAIN RESULTS

Lemma 2.1
Every mapping in OCT_n are contraction while every mapping in OC^*T_n are nonexpansive in a complete metric space (Ξ, ρ) for $\Xi \subset \mathbb{R}$.

The following examples are given to justify Lemma 2.1.

Example 1
Let S be a semigroup, $\chi_\eta = \{1, 2, \cdots, \eta\}$ a finite set ordered in standard way, ϑ: $\chi_\eta \to \chi_\eta$ be a transformation of ϑ and OCT_n a subsemigroup of order preserving full contraction mapping. Consider the transformation ϑ: $\{1, 2\} \to \{1, 2\}$ defined by ϑ = (1)(2) in OCT_2 corresponding to fixed point problem $\mathbb{T}\mu = \mu^2 - 2\mu + 2$ in a complete metric space (Ξ, ρ). Obviously, ϑ is a contraction mapping by virtue of (2). Now, we show that $\mathbb{T}$ satisfies (4) of Definition 1.1. Since (Ξ, ρ) is a complete metric space and define$\mathbb{T}$: $(\frac{1}{2}, \frac{3}{2}) \to (\frac{1}{2}, \frac{3}{2})$ 2 a self-map then, by definition

$\rho(\mathbb{T}\mu, \mathbb{T}\nu) = |\mu^2 - 2\mu + 2 - \nu^2 + 2\nu - 2|$
$= |[(\mu + \nu) - 2](\mu - \nu)|$
$\leq |[(\mu + \nu) - 2]|\, |(\mu - \nu)|$
$\leq \kappa\rho(\mu, \nu)$
for all $\mu, \nu \in (\frac{1}{2}, \frac{3}{2})$ and $0 < \kappa < 1$.
Hence, $\mathbb{T}$ is a contraction mapping in a complete metric space (Ξ, ρ).

Example 2
Let $\vartheta = (1)[2\ 1)(3) \subset OCT_3$ with a fixed point problem $\mathbb{T}\,\mu = \mu^2 - 3\mu + 3$. Then, given a self-map $\mathbb{T} : [1, 2] \to [1, 2]$ endowed with usual metric and $\mu << \nu$ since $\mu, \nu \in OC^{*}T_{\eta}$..
Clearly, ϑ is contractive in $OC^{*}T_3$ and satisfy (3). Then,
$\rho(\mathbb{T}\mu, \mathbb{T}\nu) = |\mu^2 - 3\mu + 3 - \nu^2 + 3\nu - 3|$
$= |[(\mu + \nu) - 3]\,(\mu - \nu)|$
$\leq |[(\mu + \nu) - 3]|\, |(\mu - \nu)|$
$\leq \rho(\mu, \nu)$.
for all $\mu, \nu \in [1, 2]$.
So, $\mathbb{T}$ satisfies (5) and hence, nonexpansive.

Lemma 2.2
Let (Ξ, ρ) be a metric space, then every sequence $\{\mu_\eta\}$ corresponding to $\mathbb{T} \subset OCT_{\eta}$ is monotone increasing and hence convergent in Ξ.

Theorem 2.3
Let (Ξ, ρ) be a complete metric space, $\mathbb{T}$ a corresponding self map of order preserving full contraction satisfying

$$\rho(\mathbb{T}\mu, \mathbb{T}\nu) \leq \omega_T^{\pm}(\mu, \nu) - \zeta(\omega_T^{\pm}(\mu, \nu)) \quad (6)$$

where

$$\omega_T^{\pm}(\mu, \nu) = \max\{\rho(\mu, \nu), \rho(\mu, \mathbb{T}\mu), \rho(\nu, \mathbb{T}\nu), \tfrac{1}{2}[\rho(\mu, \mathbb{T}\nu), \rho(\nu, \mathbb{T}\mu)]\} \quad (7)$$

together with ζ as stated in Theorem 1.5$\forall \mu, \nu \in \Xi$.
Then $\mathbb{T}$ has a unique fixed point.

Proof.

For any $\mu_o \in \Xi$ and sequence $\{\mu_\eta\}$, we define an iterative sequence as follows; $\mu_\eta = \mathbb{T}\mu_{\eta-1}$.
If there is a positive integer σ such that $\mathbb{T}\sigma = \sigma$, then σ is a fixed point of $\mathbb{T}$. Let assume that $\mu_{\eta-1} \neq \mu_\eta$ for all $\eta \geq 1$.
Then, from condition (6) together with (7),

$$\begin{aligned}\rho(\mu_\eta,\mu_{\eta+1}) &= \rho(\mathbb{T}\mu_{\eta-1},\mathbb{T}\mu_\eta)\\ &\leq \omega_T^{\pm}(\mu_{\eta-1},\mu_\eta) - \zeta(\omega_T^{\pm}(\mu_{\eta-1},\mu_\eta))\\ &= \max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1}), \tfrac{1}{2}[\rho(\mu_{\eta-1},\mu_{\eta+1}) + \rho(\mu_\eta,\mu_\eta)]\} - \zeta(\max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_{\eta-1},\mu_\eta)\ \rho(\mu_\eta,\mu_{\eta+1}), \tfrac{1}{2}[\rho(\mu_{\eta-1},\mu_{\eta+1} + \rho(\mu_\eta,\mu_\eta)]\})\\ &= \max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1}), \tfrac{1}{2}[\rho(\mu_{\eta-1},\mu_{\eta+1})]\}\\ &\quad - \zeta(\max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1}), \tfrac{1}{2}[\rho(\mu_{\eta-1},\mu_{\eta+1})]\}).\end{aligned}$$

But, $\frac{1}{2}[\rho(\mu_{\eta-1},\mu_{\eta+1})] \leq \frac{1}{2}[\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1})] \leq \max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1})]\}$
Then we have

$$\begin{aligned}\rho(\mu_\eta,\mu_{\eta+1}) \leq\ & (\max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1}), \max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1})\}\}) -\\ & \zeta(\max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1}), \max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1})\}\})\\ =\ & (\max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1})\})\\ & - \zeta(\max\{\rho(\mu_{\eta-1},\mu_\eta), \rho(\mu_\eta,\mu_{\eta+1})\})\end{aligned}$$

Since the self-map $\mathbb{T}$ is defined in OCT_η and by virtue of Lemma 2.2,
$\rho(\mu_{\eta-1},\mu_\eta) \leq \rho(\mu_\eta,\mu_{\eta+1})$

So, we obtain
$\rho(\mu_\eta,\mu_{\eta+1}) \leq \rho(\mu_\eta,\mu_{\eta+1}) - \boldsymbol{\zeta}(\rho(\mu_\eta,\mu_{\eta+1}))$.
i.e. $\boldsymbol{\zeta}(\rho(\mu_\eta,\mu_{\eta+1})) \leq 0$ implies that $\rho(\mu_\eta,\mu_{\eta+1}) = 0$ which means $\mu_\eta = \mu_{\eta+1}$ which is a contradiction.

, Also, by Lemma 2.2, $\rho(\mu_{\eta-1},\mu_\eta) \leq \rho(\mu_\eta,\mu_{\eta+1})$ for all $\eta \geq 1$ then there exist ð≥0 such that
$\rho(\mu_{\eta-1},\mu_\eta) = ð$ as $\eta \to \infty$.
Now, we have for all $\eta \geq 1$
$\rho(\mu_{\eta-1},\mu_\eta) \leq \rho(\mu_{\eta-1},\mu_\eta) - \boldsymbol{\zeta}(\rho(\mu_{\eta-1},\mu_\eta))$
Taking the limit as $\eta \to \infty$ in the above inequality and using the continuity of ζ, we have
(ð) ≤ (ð) − ζ (ð),
which implies that ð = 0.
Hence, $\rho\,(\mu_{\eta-1}, \mu_\eta) \to 0$ as $\eta \to \infty$

Suppose $\{\mu_\eta\}$ is not a Cauchy sequence, then for any $\epsilon>0$, we can find monotone nondecreasing sequences with $\varsigma_{(\kappa)} > \kappa$ and $\eta_{(\kappa)} > \kappa$ for all positive integers κ such that:

$$\rho(\mu_{\varsigma(\kappa)},\mu_{\eta(\kappa)}) > \epsilon \quad (8)$$

$\rho(\mu_{\varsigma(\kappa)},\mu_{\eta(\kappa)-1}) \leq \epsilon$ (9)
for $\varsigma(\kappa) > \eta(\kappa) > \kappa$.

From (8) and (9) and by triangle inequality we have

$\epsilon \leq \rho(\mu_{\varsigma(\kappa)},\mu_{\eta(\kappa)}) \leq \rho(\mu_{\varsigma(\kappa)},\mu_{\eta(\kappa)-1})$
$+ \rho(\mu_{\eta(\kappa)-1}\mu_{\eta(\kappa)})$
$< \epsilon + \epsilon$

Letting $\kappa \to \infty$ in the above inequality and by Lemma 1.6 gives
$\lim_{\kappa\to\infty}\rho(\mu_{\varsigma(\kappa)},\mu_{\eta(\kappa)}) = \epsilon$.

Also for all $k \geq 1$,

$\rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)-1}) \leq \rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)}) + \rho(\mu_{\varsigma(\kappa)},\mu_{\varsigma(\kappa)-1})$
and
$\rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)}) \leq \rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)-1}) + \rho(\mu_{\varsigma(\kappa)-1},\mu_{\varsigma(\kappa)})$.

Taking the limit as $\kappa \to \infty$ in the above inequalities and by Lemma 1.6, we have
$\lim_{\kappa\to\infty}\rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)-1}) = \epsilon$ *(10)*

Again for $k \geq 1$,

$\rho(\mu_{\eta(\kappa)+1},\mu_{\varsigma(\kappa)}) \leq \rho(\mu_{\eta(\kappa)+1},\mu_{\eta(\kappa)}) + \rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)})$
and
$\rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)}) \leq \rho(\mu_{\eta(\kappa)},\mu_{\eta(\kappa)+1}) + \rho(\mu_{\eta(\kappa)+1},\mu_{\varsigma(\kappa)})$.

Taking the limit as $\kappa \to \infty$ in the above inequalities and by Lemma 1.6, gives

$\lim_{\kappa\to\infty}\rho(\mu_{\eta(\kappa)+1},\mu_{\varsigma(\kappa)}) = \epsilon$ *(11)*
From (6) and (7), it becomes

$\rho(\mu_{\varsigma(\kappa)},\mu_{\eta(\kappa)+1}) = \rho(\mathbb{T}\mu_{\varsigma(\kappa)-1},\mathbb{T}\mu_{\eta(\kappa)})$
$\leq \boldsymbol{\omega_T^{\ddagger}}\rho(\mu_{\varsigma(\kappa)-1},\mu_{\eta(\kappa)}) - \boldsymbol{\zeta}\ (\boldsymbol{\omega_T^{\ddagger}}\rho(\mu_{\varsigma(\kappa)-1},\mu_{\eta(\kappa)}))$
$= \max\{\rho(\mu_{\varsigma(\kappa)-1},\mu_{\eta(\kappa)}),\rho(\mu_{\varsigma(\kappa)-1},\mu_{\varsigma(\kappa)}),$
$\rho(\mu_{\eta(\kappa)},\mu_{\eta(\kappa)+1}),\ \frac{1}{2}[\rho(\mu_{\varsigma(\kappa)-1},\mu_{\eta(\kappa)+1}),$
$\rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)}) - \boldsymbol{\zeta}\quad (\max\{\rho(\mu_{\varsigma(\kappa)-1},\mu_{\eta(\kappa)}),\rho(\mu_{\varsigma(\kappa)-1},\mu_{\varsigma(\kappa)}),\ \rho(\mu_{\eta(\kappa)},\mu_{\eta(\kappa)+1}),\ \frac{1}{2}[\rho(\mu_{\varsigma(\kappa)-1},\mu_{\eta(\kappa)+1}),$
$\rho(\mu_{\eta(\kappa)},\mu_{\varsigma(\kappa)})]$

Let $\kappa \to \infty$ in the above inequality, then by Lemma 1.6 and by (10), (11) and also by virtue of ζ, we have

$\epsilon = \epsilon - \zeta(\epsilon)$

this implies that, $\zeta(\epsilon) = 0$. Hence, μ_η is a Cauchy sequence. Also, there exists $\sigma \in \Xi$ such that:

$\mu_\eta \to \sigma$ as $\eta \to \infty$

Also from (6) and (7) we have

$\rho(\mu_{\eta+1},\mathbb{T}\sigma) = \rho(\mathbb{T}\mu_\eta,\mathbb{T}\sigma)$
$\leq \boldsymbol{\omega_T^{\ddagger}}(\rho(\mathbb{T}\mu_\eta,\sigma)) - \boldsymbol{\zeta}\ (\boldsymbol{\omega_T^{\ddagger}}(\rho(\mathbb{T}\mu_\eta,\sigma)))$
$= \max\{\rho(\mu_\eta,\sigma),\rho(\mu_\eta,\mu_{\eta+1}),\rho(\sigma,\mathbb{T}\sigma),\ \frac{1}{2}[\rho(\mu_\eta,\mathbb{T}\sigma)+\rho(\sigma,\mu_{\eta+1})]\} - \zeta(\max\{\rho(\mu_\eta,\sigma),\rho(\mu_\eta,\mu_{\eta+1}),\rho(\sigma,\mathbb{T}\sigma),\ \frac{1}{2}[\rho(\mu_\eta,\mathbb{T}\sigma)+\rho(\sigma,\mu_{\eta+1})]\})$

Taking the limit as $\eta \to \infty$ in the above inequality and using the continuity of ζ, we have

$\rho(\sigma,\mathbb{T}\sigma) = \rho(\sigma,\mathbb{T}\sigma) - \zeta\ (\rho(\sigma,\mathbb{T}\sigma))$.

Which means, $\rho(\sigma,\mathbb{T}\sigma) = 0$ i.e $\sigma = \mathbb{T}\sigma$. Hence *$\sigma$ is a fixed point of* $\mathbb{T}$.

Let *σ and σ^* be two fixed points of* $\mathbb{T}$ and suppose that $\sigma \neq \sigma^*$*,then, from (6) together with (7) we have*

$\rho(\mathbb{T}\sigma,\mathbb{T}\sigma^*) = \rho(\sigma,\sigma^*) \leq (\omega_T^{\ddagger}(\rho(\sigma,\sigma^*))$
$- \zeta(\omega_T^{\ddagger}(\rho(\sigma,\sigma^*))$
$= \max\{\rho(\sigma,\sigma^*),\rho(\sigma,\mathbb{T}\sigma),\rho(\sigma^*,\mathbb{T}\sigma^*),\ \frac{1}{2}[\rho(\sigma,\mathbb{T}\sigma^*)+\rho(\sigma^*,\mathbb{T}\sigma)]\} - \zeta(\max\{\rho(\sigma,\sigma^*),\rho(\sigma,\mathbb{T}\sigma),\rho(\sigma^*,\mathbb{T}\sigma^*),\ \frac{1}{2}[\rho(\sigma,\mathbb{T}\sigma^*) + \rho(\sigma^*,\mathbb{T}\sigma)]\}$
$= \max\{\rho(\sigma,\sigma^*),\ \rho(\sigma,\sigma),\ \rho(,\sigma^*,\sigma^*),$
$\frac{1}{2}[\rho(\sigma,\sigma^*) + \rho(\sigma,\sigma^*)]\} - \zeta(\max\{\rho(\sigma,\sigma^*),\ \rho(\sigma,\sigma),\ \rho(,\sigma^*,\sigma^*),\ \frac{1}{2}[\rho(\sigma,\sigma^*) + \rho(\sigma^*,\sigma)]\})$
$= \rho(\sigma,\sigma^*) - \zeta(\rho(\sigma,\sigma^*))$

i.e $\rho(\sigma,\sigma^*) \leq \rho(\sigma,\sigma^*) - \zeta\ (\ \rho(\sigma,\sigma^*))$ which implies $\zeta(\rho(\sigma,\sigma^*) \leq 0$

which is a contradiction by virtue of $\boldsymbol{\zeta}$. Hence $\sigma = \sigma^*$.

Corollary 2.4

Let (Ξ, ρ) be a complete metric space and $\mathbb{T} : \Xi \to \Xi$ a self map deduced from generalized order preserving full contraction mapping such that for all $\mu, \nu \in \Xi$,

$$\varpi(\rho(\mathbb{T}\mu, \mathbb{T}\nu)) \leq \varpi(\omega_T^{\ddagger}(\mu,\nu)) - \zeta(\omega_T^{\ddagger}(\mu,\nu)) \quad (12)$$

Where $\omega_T^{\ddagger}(\mu,\nu)$ is as defined in (7). Then, there exists a unique fixed point.

Proof: Letting $\varpi = I$, the identity mapping on Ξ and follow the analogue of Theorem 2.3 then, the result follows.

Example 3. Let ϑ be a transformation in OCT_η define $\vartheta : \{1,2,3\} \to \{1,2,3\}\ by\ \vartheta = (1)[321)$ which is equivalent to the fixed point problem

$\mathbb{T}\mu = \frac{\mu 2-3\mu+4}{2}$ with $F_{\mathbb{T}} = \{1\}$, where $\mathbb{T}$ is a self map defined as $\mathbb{T}: [1, 2] \to [1, 2]$. Also let Ξ be endowed with the usual metric $\rho(\mu, \nu) = |\mu - \nu|$ and define $\varpi(\tau) = \frac{\tau^2}{2}$ and $\zeta(\tau) = a\tau \, for \; 0 < a < \frac{1}{2}; 0 \le \tau \le 1; \mu, \nu \in \Xi$. Obviously, $\mathbb{T}$ is a contraction mapping for $\rho(\mathbb{T}\mu, \mathbb{T}\nu)) \le k(\rho(\mu, \nu))$ and $0 < k < 1$. Now let $\mu = 1, \nu = 2$ and $a = \frac{1}{4}$, then (7) $\omega_T^{\ddagger}(\mu, \nu) = \max\{1, 0, 1, \frac{1}{2}, (0+1)\}$ and $\varpi(0) = 0, \varpi(1) = \frac{1}{2}$ and $\zeta(1) = \frac{1}{4}$. Thus, $\mathbb{T}$ satisfies contractive condition (12).

More so, choose $\mu_0 = 1.1$ so that

$\mathbb{T}\mu_0 = \left|\frac{4(1.1)-6}{4}\right| < 1$. This implies that, $\mathbb{T}$ converges. Then, define an iterative sequence $\mu_{\eta+1} = \mathbb{T}\,\mu_0, \eta \ge 0$ so that $\mu_1 = \mathbb{T}\,\mu_0 = 1.055, \dots\dots, \mu_{12} = \mathbb{T}\,\mu_{12} \approx 1.0000$(4dp).

This shows that $\mathbb{T}$ has a unique fixed point which is comparable with the fixed elements of $\vartheta \in OCT_\eta$.

Remark 2.5

Contractive condition (6) and (12) with respect to (7) holds for Example 3 when $\mathbb{T}$ is defined in OCT_η but fails when define in $OC{*}T_\eta$.

Theorem 2.6

Let (Ξ, ρ) be a complete metric space and $\mathbb{Q}, \mathbb{T} : \Xi \to \Xi$ be two corresponding self maps in OCT_n and $OC{*}T_n$ such that for all $\mu, \nu \in \Xi$,

$$\varpi(\rho(\mathbb{Q}\mu, \mathbb{T}\nu)) \le \varpi(\omega_T^{\ddagger}(\mu, \nu)) - \zeta(\omega_T^{\ddagger}(\mu, \nu)) . \quad (13)$$

Where $\omega_T^{\ddagger}(\mu, \nu)$ is as defined in (7).

Then, there exists a unique common fixed point say $\sigma \in \Xi$ such that $\sigma = \mathbb{Q}\sigma = \mathbb{T}\sigma$.

Proof. Let $\mu_0 \in \Xi$ be arbitrary point, we construct the sequence $\{\mu_\eta\}$ such that μ_η in Ξ so that $\mu_{2\eta+1} = \mathbb{Q}\,\mu_{2\eta}$ and $\mu_{2\eta+2} = \mathbb{T}\,\mu_{2\eta+1}$ for $\eta \ge 0$.

Now, from (7) we have $\omega_T^{\ddagger}(\mu_{2\eta+1}, \mu_{2\eta+2})$

$=\max\{\rho(\mu_{2\eta+1}, \mu_{2\eta+2}), \rho(\mu_{2\eta+1}, \mathbb{Q}\,\mu_{2\eta}), \rho(\mu_{2\eta+2}, \mathbb{T}\mu_{2\eta+1}), \frac{1}{2}[\rho(\mu_{2\eta+1}, \mathbb{T}\mu_{2\eta+1}) + \rho(\mu_{2\eta+2}, \mathbb{Q}\mu_{2\eta})]\}$

$=\max\{\rho(\mu_{2\eta+1}, \mu_{2\eta+2}), \rho(\mu_{2\eta+1}, \mu_{2\eta+1}), \rho(\mu_{2\eta+2}, \mu_{2\eta+2}), \frac{1}{2}[\rho(\mu_{2\eta+1}, \mu_{2\eta+2}) + \rho(\mu_{2\eta+2}, \mu_{2\eta+1})]\}$

$=\max\{\rho(\mu_{2\eta+1}, \mu_{2\eta+2}), \frac{1}{2}[\rho(\mu_{2\eta+1}, \mu_{2\eta+2})]\}$

$=\rho(\mu_{2\eta+1}, \mu_{2\eta+2})$

Now, from (13) we have

$\varpi\left(\rho(\mu_{2\eta+1}, \mu_{2\eta+2})\right) = \varpi\left(\rho(\mathbb{Q}\,\mu_{2\eta}, \mathbb{T}\mu_{2\eta+1})\right) \le \varpi(\omega_T^{\ddagger}(\rho(\mu_{2\eta+1}, \mu_{2\eta+2}))) - \zeta(\omega_T^{\ddagger}(\rho(\mu_{2\eta+1}, \mu_{2\eta+2})))$

By virtue of properties ϖ and ζ, $\rho(\mu_{2\eta+1}, \mu_{2\eta+2}) = 0$ then, by analoque of Theorem 2.3, the sequence $\{\mu_\eta\}$ converges and also a Cauchy sequence.

So, there exist $\sigma \in \Xi$ such that $\mu_\eta \to \infty$, also $\mu_{2\eta} \to \sigma$ and $\mu_{2\eta+1} \to \sigma$. Again we show that $\mathbb{Q}\sigma = \sigma = \mathbb{T}\sigma$. Suppose $\sigma \ne \mathbb{Q}\sigma$ such that $\rho(\sigma, \mathbb{Q}\sigma) > 0$, then there exist $\eta \in \mathbb{N}$ such that

$\rho(\mu_{2\eta+1}, \sigma) < \frac{1}{2}[\rho(\sigma, \mathbb{Q}\sigma), \rho(\mu_{2\eta}, \sigma)] < \frac{1}{2}[\rho(\sigma, \mathbb{Q}\sigma), \rho(\mu_{2\eta}, \mu_{2\eta+1})] < \frac{1}{2}[\rho(\sigma, \mathbb{Q}\sigma)]$

Now,

$\rho(\mathbb{Q}\sigma, \sigma) \le \omega_T^{\ddagger}\,\rho(\sigma, \mu_{2\eta})$

$=\max\{\rho(\sigma, \mu_{2\eta}), \rho(\mathbb{Q}\sigma, \sigma), \rho(\mathbb{T}\mu_{2\eta}, \mu_{2\eta}), \frac{1}{2}[\rho(\mu_{2\eta}, \mathbb{Q}\sigma) + \rho(\sigma, \mathbb{T}\,\mu_{2\eta})]\}$

$=\max\{\rho(\sigma, \mu_{2\eta}), \rho(\mathbb{Q}\sigma, \sigma), \rho(\mu_{2\eta+1}, \mu_{2\eta}), \frac{1}{2}[\rho(\mu_{2\eta}, \mathbb{Q}\sigma) + \rho(\sigma, \mu_{2\eta+1})]\}$

$\le\max\{\frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma), \rho(\mathbb{Q}\sigma, \sigma) \; \frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma), \frac{1}{2}[\rho(\mu_{2\eta}, \mathbb{Q}\sigma) + \frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma)]\}$

$\le\max\{\frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma), \rho(\mathbb{Q}\sigma, \sigma) \; \frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma), \frac{1}{2}[\rho(\mu_{2\eta}, \sigma) + \rho(\sigma, \mathbb{Q}\sigma) + \frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma)]\}$

$\le\max\{\frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma), \rho(\mathbb{Q}\sigma, \sigma) \; \frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma), \frac{1}{2}[\frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma) + \rho(\sigma, \mathbb{Q}\sigma) + \frac{1}{2}\rho(\mathbb{Q}\sigma, \sigma)]\}$

$=\rho(\mathbb{Q}\sigma, \sigma)$

But, $\varpi\left(\rho(\mathbb{Q}\sigma, \mu_{2\eta+1})\right) = \varpi\left(\rho(\mathbb{Q}\sigma, \mathbb{T}\mu_{2\eta})\right)$

$$\le \varpi(\omega_T^{\ddagger}\left(\rho(\sigma, \mu_{2\eta})\right) - \zeta(\omega_T^{\ddagger}\left(\rho(\sigma, \mu_{2\eta})\right)$$

Then, as $\eta \to \infty$ in the above inequality, we have

$$\varpi\big(\rho(\mathbb{Q}\sigma, \sigma)\big) \le \varpi\big(\rho(\mathbb{Q}\sigma, \sigma)\big) - \zeta\big((\mathbb{Q}\sigma, \sigma)\big)$$

Which is a contradiction unless thus

$\rho(\mathbb{Q}\sigma, \sigma) = 0$. Hence $\mathbb{Q}\sigma = \sigma$

Furthermore, for $\mathbb{Q}\sigma = \mathbb{T}\sigma = \sigma$

$$\rho(\mathbb{T}\sigma, \sigma) \le \omega_T^{\ddagger}(\sigma, \sigma)$$
$$= \max\{\rho(\sigma, \sigma), \rho(\mathbb{Q}\sigma, \sigma). \rho(\mathbb{T}\sigma, \sigma).$$

$$\frac{1}{2}[\rho(\sigma,\mathbb{Q}\sigma)+\rho(\sigma,\mathbb{T}\sigma)]\}$$

$=\max\{0,0,\ \rho(\mathbb{T}\sigma,\sigma).\frac{1}{2}[0+\rho(\sigma,\mathbb{T}\sigma)]\}$

$=\rho(\mathbb{T}\sigma,\sigma)$

Therefore, $\omega_T^{\ddagger}(\sigma,\sigma)=\rho(\mathbb{T}\sigma,\sigma)$.

Since,$\varpi\big(\rho(\sigma,\mathbb{T}\sigma)\big)=\varpi\big(\rho(\mathbb{Q}\sigma,\mathbb{T}\sigma)\big)\le$ $\varpi\big(\omega_T^{\ddagger}(\sigma,\sigma)\big)-\zeta\big(\omega_T^{\ddagger}(\sigma,\sigma)\big)$

$=\varpi\big(\rho(\sigma,\mathbb{T}\sigma)\big)-\zeta\big(\rho(\sigma,\mathbb{T}\sigma)\big)$

Then $\rho(\sigma,\mathbb{T}\sigma)=0$ or $\mathbb{T}\sigma=\sigma$

By virtue of ϖ and ζ. Hence, $\mathbb{Q}\sigma=\sigma=\mathbb{T}\sigma$

Suppose there exists another point $\sigma^*\in\Xi$ such that $\mathbb{Q}\sigma^*=\sigma^*=\mathbb{T}\sigma^*$, then using an argument similar to the above we have,

$$\varpi\big(\rho(\sigma,\sigma^*)\big)=\varpi\big(\rho(\mathbb{T}\sigma,\mathbb{T}\sigma^*)\big)$$
$$\le\varpi\big(\omega_T^{\ddagger}(\sigma,\sigma^*)\big)-\zeta\big(\omega_T^{\ddagger}(\sigma,\sigma^*)\big)$$
$$=\varpi\big((\sigma,\sigma^*)\big)-\zeta\big((\sigma,\sigma^*)\big)$$

Hence, $\sigma=\sigma^*$.

Corollary 2.7

Let (Ξ,ρ) be a complete metric space and $\mathbb{Q}$, $\mathbb{T}$:

$\Xi\to\Xi$ be two self maps such that for all $\mu,\ v\in\Xi$

$$\rho(\mathbb{Q}\mu,\mathbb{T}v)\le\big(\omega_T^{\ddagger}(\mu,v)\big)-\zeta(\omega_T^{\ddagger}(\mu,v))$$

Where $\omega_T^{\ddagger}(\mu,v)$ is dfined in (7). Then, $\mathbb{Q}$,and $\mathbb{T}$ have a unique common fixed point.

Proof. Making $\varpi=I$ in the first and second of Theorem 2.6 and follow its analogue then, the result follows.

Example 4.

Consider the mappings $\vartheta=(1)[2\ 1)(3)\subset OC^*T_3$ and $\delta=(1)[3\ \ 2\ \ 1)\subset OCT_3$ with fixed point problems $\mathbb{Q}\mu=\mu^2-3\mu+3$ and $\mathbb{T}v=\frac{v^2-3v+4}{2}$ respectively in a complete metric space (Ξ, ρ)

Where $\mathbb{Q}$,, $\mathbb{T}$ are two self maps defined as

$\mathbb{Q},\mathbb{T}:[1,2]\to[1,2]$ in (Ξ, ρ). Thus, we show that $\mathbb{Q}\ and\ \mathbb{T}$ satisfies condition (13) with respect to $\omega_T^{\ddagger}(\mu,v)$ stated in (7). Since $\mu<v$, now let $\mu=1$ and $v=2$ so that $\rho(\mu,v)=1$, $\rho(\mu,\mathbb{Q}\mu)=0$, $\rho(v,\mathbb{T}v)=1$, $\rho(\mu,\mathbb{Q}v)=0$

$\rho(v,\mathbb{Q}\mu)=1$ and $\rho(\mathbb{Q}\mu,\mathbb{T}v)=0$. Again, define $\varpi(\tau)$ and $\zeta(\tau)$ as given in Example 3 we have $\varpi(0)=0$, $\varpi(1)=\frac{1}{2}$ and $\zeta(1)=\frac{1}{4}$. Clearly, the two mappings satisfies the contractive conditions (12) and (13). Hence the deterministic fixed points of both $\mathbb{Q}$ and $\mathbb{T}$ are comparable with the fixed elements $\vartheta\in OC^*T_3$ and $\delta\in OCT_3$.

Remark 2.8

The contractive conditions (6), (12) and (13) with respect to (7) holds for any two associated mappings in both *OCT* $_\eta$ and OC^*T_η.

CONCLUSION

In this paper, we have shown that, every images of ϑ which are fixed under the action of the subsemigroups OC^*T_η and OCT_η at certain point are unique and comparable with the deterministic fixed points in a complete metric space (Ξ, ρ). Also, $\mathbb{T}$ is non expansive in metric space if the transformation ϑ is contractive that is if $\vartheta\subset OC^*T_\eta$ and $\mathbb{T}$ is closed otherwise it is *L*-Lipchisctian for $L>1$as discussed in Example 2.

More so, the common fixed elements of any two transformations on these subsemigroups are comparable with the approximate fixed points of their correspondence two self maps in a complete metric space as analyzed in Example 4. The key feature in these fixed point theorems is that, the contractivity condition on the self map is weakly and only assumed to hold on elements of subsemigroups of order preserving full contraction and contractive mapping in metric spaces.

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