Shehu Transform Homotopy Analysis Method for the Solution of Nonlinear Initial Value Ordinary and Partial Differential Equations

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Abstract: In this paper, we propose a method of solution for both nonlinear Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). Shehu Transform is combined with Homotopy Analysis Method (HAM) to handle both homogeneous and nonhomogeneous problems in the family of differential equations considered. The properties of homotopy derivatives are exploited while handling the nonlinear terms encountered. Several examples are solved using HAM and the Shehu Transform Homotopy Analysis Method (STHAM) proposed in this work, and the effectiveness of the proposed method is obvious in terms of reduction in the volume of computations and time. All computations are carried out with the aid of Mathematica 12.0.

Keywords: *Shehu Transform, Homotopy Derivatives, Embedding Parameter, Control Parameter, Deformation Equations.*

1. INTRODUCTION

Larger percentage of physical phenomena, when modelled result into nonlinear ordinary or partial differential equations. The differential equations thus formed are mostly very difficult, if not impossible to solve using the traditional analytical approaches, and that informed the development of numerical methods that often give approximate solutions. The need for more reliable results brought about the development of semi analytical methods that generate exact solutions whenever such exist in closed form. Prominent among such methods are the Adomian Decomposition Method (ADM) introduced by Gorge Adomian in the early 1990s, Homotopy Analysis Method (HAM) [1], Homotopy Perturbation Method (HPM) [2], Variational Iteration Method (VIM) [3], just to mention a few.

Homotopy Analysis Method makes use of two deformation equations which are referred to as the zeroth order and nth order deformation equations. These equations generate improved results that eventually get closer to the exact solutions within limited number of iterations ([4], [5], [6]).

Integral transforms such as Laplace [7], Elzaki [8], Aboodh [9], Sumudu [10] and Shehu [6], just to mention a few, are very useful methods in solving initial value problems (IVPs) in both ordinary differential equations (ODEs) and partial differential equations (PDEs), although some have limitations to constant coefficients cases. Meanwhile, the choice of Shehu transform in our earlier work [11] informed by two special qualities that Shehu transform possesses, which are, that; it solves constant coefficients IVPs as well as variable coefficients ones, and the fact that it generalizes the earlier transforms such as Laplace and Sumudu ([7], [12]). Handibag and Wayal applied some other methods to obtain the solution of system of nonlinear PDEs [17].

In this work therefore, Shehu transform is combined with Homotopy analysis method to solve nonlinear initial value problems in both ordinary differential equations and partial differential equations. The proposed method is easy to implement, and it reduces computations in terms of space and time.

2. STATEMENT OF THE PROBLEM

The class of problem considered in the research is the general nonlinear differential equation

$$Lu(x,y) + N\big(u(x,y)\big) + Qu(x,y) = f(x,y),$$

where $L, N, Q,$ and f are respectively the linear, nonlinear, the remaining linear terms and the inhomogeneous source term.

This family of problem above is solved using Homotopy Analysis Method (HAM) and Shehu Transform Homotopy Analysis Method (STHAM) the details of which are presented in the sequel.

3. PRELIMINARIES

A. Homotopy Analysis Method of Solving Nonlinear Differential Equations [13]

Consider the nonlinear initial value differential equation

$$Lu(x) + N\big(u(x)\big) = f(x),$$

where L is the linear operator, N is the nonlinear operator and $f(x)$ is an inhomogeneous source term.

There are two deformation equations: the zeroth order deformation equation and the nth order deformation equation which is derived from zeroth order after differentiating it n times with respect to the embedding parameter q, and setting $q = 0$. The zeroth deformation equation takes the form

$$(1-q)L[(\emptyset(x;q)-\emptyset(x;0)] = qhN[\emptyset(x;q)],$$

while the nth order deformation takes the form

$$L[u_n(x) - \chi_n u_{n-1}(x)] = hD_{n-1}[N[\phi(x;q)],$$

where

$$\chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases}$$

$q \in [0,1]$ is the embedding parameter, h is the control parameter and

$$D_{n-1}[N[\phi(x;q)] = \frac{1}{(n-1)!}\frac{\partial^{n-1}[N[\phi(x;q)]}{\partial q^{n-1}}$$

is the homotopy derivative.

The rules guiding homotopy derivatives include:

$$D_n(\emptyset^2) = \sum_{j=0}^{n} D_j(\emptyset)\, D_{n-j}(\emptyset) = \sum_{j=0}^{n} u_j u_{n-j}$$

and

$$D_n(\emptyset^3) = \sum_{j=0}^{n} D_j(\emptyset)\, D_{n-j}(\emptyset^2) = \sum_{j=0}^{n} u_{n-j} \sum_{k=0}^{j} u_k u_{j-k}.$$

For details on HAM, interested readers can check (Liao, 2009).

B. **Shehu Transform for Derivatives**

The Shehu transform of the term $u^{(n)}(x)$, where n is the order of derivative and $n \in \mathbb{N}$ is given as

$$S\{u^{(n)}(x)\} = \frac{s^n}{v^n}U(s,v) - \sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0)$$

such that

$$S\{u(x)\} = U(s,v)$$

and

$$S\{u'(x)\} = \frac{s}{v}U(s,v) - u(0).$$

4. **Application of Homotopy Analysis Method to Nonlinear Initial Value Problems in ODEs**

Problem 1 [14]

Solve the second order nonlinear IVP below using HAM

$$y''(x) + 2y(x)y'(x) = 0, \quad y(0) = 0, y'(0) = 1. \qquad (i)$$

Solution

The initial approximation

$y_0(x) = y(0) + xy'(0) = 0 + x.1 = x.$

The auxiliary linear operator

$L[y(x)] = y''(x),$

while the nonlinear operator

$$N[y(x)] = y''(x) + 2y(x)y'(x).$$

Using the nth order deformation equation

$$L[y_n(x) - \chi_n y_{n-1}] = hD_{n-1}[y''(x) + 2y(x)y'(x)]],$$

where

$$\chi_n = \begin{cases} 0, & for\ n \le 1 \\ 1, & for\ n > 1 \end{cases}$$

and

$$D_{n-1}[N[y(x)]] = D_{n-1}[y''(x) + 2y(x)y'(x)]]$$

$$= y''_{n-1}(x) + 2\sum_{j=0}^{n-1} y_j(x)y'_{n-1-j}(x)$$

For $n = 1$:

$$D_0[N[y(x)]] = y''_{n-1}(x) + 2\sum_{j=0}^{0} y_j(x)y'_{n-1-j}(x)$$
$$= y''_0(x) + 2y_0(x)y'_0(x)$$

For $n = 2$:

$$D_1[N[y(x)]] = y''_1(x) + 2\sum_{j=0}^{1} y_j(x)y'_{n-1-j}(x)$$
$$= y''_0(x) + 2y_0(x)y'_0(x)$$

For $n = 3$:

$$D_2[N[y(x)]] = y''_2(x) + 2\sum_{j=0}^{2} y_j(x)y'_{n-1-j}(x)$$
$$= y''_2(x) + 2(y_0(x)y'_2(x) + y_1(x)y'_1(x) + y_2(x)y'_0(x))$$

And so on.

Using the above derivations in the nth order deformation equation, we have

$$L[y_1(x) - \chi_1 y_0] = hD_0N[y(x)]$$

$$y''_1(x) = h(y''_0(x) + 2y_0(x)y'_0(x))$$

$$y''_1(x) = h(0 + 2(x.1))$$
$$y''_1(x) = 2hx$$

Integrating, we get

$$y'_1(x) = hx^2 + c_1$$

Using the initial conditions, we have $y_1(0) = 0$, hence $c_1 = 0$. Thus,

$y'_1(x) = hx^2.$

Integrating for the second time, we get

$$y_1(x) = \frac{1}{3}hx^3 + c_2,$$

which also gives $c_2 = 0$ upon substitution of initial values. Therefore,

$$y_1(x) = \tfrac{1}{3}hx^3.$$

The second iteration:

$$L[y_2(x) - \chi_2 y_1] = hD_1N[y(x)]$$
$$y''_2(x) - y''_1(x) = h[y''_1(x) + 2(y_0(x)y'_1(x) + y_1(x)y'_0(x))]$$
$$y''_2(x) - 2hx = h[2hx + 2(x.hx^2 + \frac{1}{3}hx^3.1)]$$

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$$y_2''(x) = 2hx + 2hx^2 + \frac{8}{3}h^2x^3$$

$$y_2'(x) = hx^2 + 2h^2x^2 + \frac{2}{3}h^2x^4 + c_1.$$

Using the initial conditions, we get $c_1 = 0$. Hence,

$$y_2'(x) = hx^2 + 2h^2x^2 + \frac{2}{3}h^2x^4.$$

$$y_2(x) = \frac{1}{3}hx^3 + \frac{1}{3}h^2x^3 + \frac{2}{15}h^2x^5 + c_2.$$

Using the initial conditions, $c_2 = 0$. Therefore,

$$y_2(x) = \frac{1}{3}hx^3 + \frac{1}{3}h^2x^3 + \frac{2}{15}h^2x^5.$$

The third iteration:

$$L[y_3(x) - \chi_3 y_2] = hD_2 N[y(x)]$$

Integrating twice and using the relevant boundary conditions, we get

$$y_3(x) = \frac{1}{3}hx^3 + \frac{2}{3}h^2x^3 + \frac{1}{3}h^3x^3 + \frac{4}{9}h^3x^4 + \frac{1}{10}h^3x^5 + \frac{1}{30}h^3x^5 + \frac{17}{315}h^3x^7 + \frac{1}{63}h^4x^7 + \frac{1}{162}h^4x^9.$$

When $h = -1$:

$$y_0(x) = x,$$

$$y_1(x) = -\frac{x^3}{3},$$

$$y_2(x) = -\frac{x^3}{3} + \frac{x^3}{3} + \frac{2x^5}{15}$$

In like manner we get the $y_3(x)$, and so on.

The solution $y(x)$ to the problem is given as

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots,$$

which gives

$$y(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{4}{105}x^7 + \frac{x^9}{162} + \cdots.$$

Application of Homotopy Analysis Method to Nonlinear Initial Value Problems in PDEs

Problem 1 [12]

Use HAM to solve the nonlinear PDE

$$u_t + u_x^2 = 0, \quad u(x, 0) = x, \quad t > 0.$$

Solution

The auxiliary linear operator and the nonlinear operators are respectively

$$L[u] = \frac{\partial u}{\partial t}, \quad N[u] = \frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2.$$

$$D_{n-1}[N[u]] = D_{n-1}[\frac{\partial u}{\partial t} + (\frac{\partial u}{\partial x})^2]$$

$$= \frac{\partial u_{n-1}}{\partial t} + \sum_{j=0}^{n-1} \frac{\partial u_j}{\partial x}\frac{\partial u_{n-1-j}}{\partial x}.$$

Using the nth order deformation equation

$$L[u_n(x,t) - \chi_n u_{n-1}(x,t)] = hD_{n-1}[N[u]]$$

$$\frac{\partial u_n}{\partial t} - \chi_n \frac{\partial u_{n-1}}{\partial t} = h\left[\frac{\partial u_{n-1}}{\partial t} + \sum_{j=0}^{n-1} \frac{\partial u_j}{\partial x}\frac{\partial u_{n-1-j}}{\partial x}\right].$$

Integrating through from 0 to t, with respect to t, we have

$$u_n = \chi_n u_{n-1} + h\int_0^t \left[\frac{\partial u_{n-1}}{\partial t} + \sum_{j=0}^{n-1} \frac{\partial u_j}{\partial x}\frac{\partial u_{n-1-j}}{\partial x}\right] dt.$$

Taking $h = -1$, we have

$$u_n = \chi_n u_{n-1} - \int_0^t \left[\frac{\partial u_{n-1}}{\partial t} + \sum_{j=0}^{n-1} \frac{\partial u_j}{\partial x}\frac{\partial u_{n-1-j}}{\partial x}\right] dt.$$

For $n = 1$:

$$u_1 = \chi_1 u_0 - \int_0^t \left[\frac{\partial u_0}{\partial t} + \frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial x}\right] dt.$$

But $u_0(x,t) = x$ from the boundary condition. Thus,

$$u_1 = 0.x - \int_0^t (0 + 1.1) dt = -t.$$

For $n = 2$:

$$u_2 = \chi_2 u_1 - \int_0^t \left[\frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial x}\frac{\partial u_0}{\partial x}\right] dt$$

$$u_2 = 1(-t) - \int_0^t [(-1) + 1.0 + 0.1] dt$$

$$u_2 = -t + t = 0.$$

All the subsequent $u_n(x,t) = 0$, for $n \geq 3$. Thus

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

$$u(x,t) = x - t + 0 + 0 + 0 + \cdots.$$

Hence,

$$u(x,t) = x - t.$$

Problem 2 [12]

Use HAM to solve the second order nonlinear partial differential equation

$$u_{xx} + \frac{1}{4}u_y^2 = u, \quad u(0,y) = 1 + y^2, \quad u_x(0,y) = 1.$$

Solution

The auxiliary linear operator is

$$L[u] = \frac{\partial^2 u}{\partial x^2},$$

while the nonlinear operator is

$$N[u] = \frac{\partial^2 u}{\partial x^2} + \frac{1}{4}(\frac{\partial u}{\partial y})^2 - u.$$

The initial approximation is obtained via the use of Taylor's theorem for two variables as follows:

$$u_0(x,y) = u(a,b) + u_x(a,b)(x-a) + u_y(a,b)(y-b) + \cdots,$$

But in this case, $a = 0$ and $b = y$, thus

$$u_0(x,y) = u(0,y) + u_x(0,y)(x-0) + u_y(0,y)(y-y)$$

$$u_0(x,y) = u(0,y) + u_x(0,y).x$$

Therefore,

$$u_0(x,y) = 1 + y^2 + 1.x = 1 + x + y^2.$$

Using the nth order deformation equation, we have

$$D_{n-1}[N[u]] = D_{n-1}[\frac{\partial^2 u}{\partial x^2} + \frac{1}{4}(\frac{\partial u}{\partial y})^2 - u]$$
$$= \frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{1}{4}\sum_{j=0}^{n-1} \frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y} - u_{n-1}.$$

Hence, the nth order deformation equation becomes

$$\frac{\partial^2 u_n}{\partial x^2} - \chi_n \frac{\partial^2 u_{n-1}}{\partial x^2} = h\left[\frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{1}{4}\sum_{j=0}^{n-1} \frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y} - u_{n-1}\right].$$

Integrating through from 0 to x with respect to x, we have

$$u_n - \chi_n u_{n-1} = h\int_0^x\int_0^x\left(\frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{1}{4}\sum_{j=0}^{n-1} \frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y} - u_{n-1}\right)dxdx$$

Putting $h = -1$, we have

$$u_n = \chi_n u_{n-1} - \int_0^x\int_0^x\left(\frac{\partial^2 u_{n-1}}{\partial x^2} + \frac{1}{4}\sum_{j=0}^{n-1} \frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y} - u_{n-1}\right)dxdx$$

For $n = 1$:

$$u_1 = \chi_1 u_0 - \int_0^x\int_0^x\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{1}{4}\frac{\partial u_0}{\partial y}\frac{\partial u_0}{\partial y} - u_0\right)dxdx$$
$$u_1 = 0.(1+x+y^2) - \int_0^x\int_0^x\left(0 + \frac{1}{4}(2y.2y) - (1+x+y^2)\right)dxdx$$
$$u_1 = \int_0^x\int_0^x (1+x)dxdx$$

Hence,

$$u_1 = \frac{x^2}{2} + \frac{x^3}{6}.$$

For $n = 2$:

$$u_2 = \chi_2 u_1 - \int_0^x\int_0^x\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{1}{4}(\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial u_0}{\partial y}) - u_1\right)dxdx$$
$$u_2 = \frac{x^2}{2} + \frac{x^3}{6} - \int_0^x\int_0^x\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6}\right)dxdx$$
$$u_2 = \frac{x^4}{24} + \frac{x^5}{120}.$$

The solution is given by

$$u(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + \cdots.$$
$$u(x,y) = 1 + x + y^2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$
$$u(x,y) = y^2 + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots).$$

Hence,

$$u(x,y) = y^2 + e^x.$$

5. Application of Shehu Transform Homotopy Analysis Method to Nonliear Partial and Ordinary Differential Equations

To present the algorithm for our proposed method, we shall consider a general nonlinear differential equation

$$Lu(x,y) + N(u(x,y)) + Qu(x,y) = f(x,y),$$

where L is the highest order linear operator, N is the nonlinear operator, Q is the remaining linear terms and $f(x,y)$ is the inhomogeneous source term which is a function of two variables x and y.

The Shehu transform shall be applied to every term of the given equation as follows

$$S\{Lu(x,y)\} + S\{N(u(x,y))\} + S\{Qu(x,y)\} = S\{f(x,y)\},$$

such that

$$S\{Lu(x,y)\} = S\left\{\frac{\partial^n y}{\partial x^n}\right\} = S\{u^{(n)}\}$$
$$S\{u^{(n)}\} = \frac{s^n}{v^n}U(s,v) - \sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0)$$

Putting (3) in (2) gives

$$\frac{s^n}{v^n}U(s,v) - \sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0) + S\{N(u(x,y))\} + S\{Qu(x,y)\} = S\{f(x,y)\}, \quad (4)$$

where the Shehu transform of $u(x,y)$ is $S\{u(x,y)\} = U((s,v),y)$ and $u^{(j)}(0,y)$ represent the sequence $u(0,y), u_x(0,y), u_{xx}(0,y), \ldots$, for $j = 0,1,2,\ldots,n-1$.

Dividing (4) through by $\frac{s^n}{v^n}$, we have

$$U(s,v) - \frac{v^n}{s^n}\sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0) + \frac{v^n}{s^n}S\{N(u(x,y))\} + \frac{v^n}{s^n}S\{Qu(x,y)\} - \frac{v^n}{s^n}S\{f(x,y)\} = 0 \quad (5)$$

From (5), the general nonlinear operator is obtained as

$$N[U((s,v),y;q)] = U(s,v) - \frac{v^n}{s^n}\sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0) + \frac{v^n}{s^n}S\{N(u(x,y))\} + \frac{v^n}{s^n}S\{Qu(x,y)\} - \frac{v^n}{s^n}S\{f(x,y)\}, \quad (6)$$

where $q \in [0,1]$ is the embedding parameter.

The results above shall be implemented in the nth order deformation equation

$$L[U_n((s,v),y) - \chi_n U_{n-1}((s,v),y)] = hD_{n-1}[N[U((s,v),y;q)]]$$

But $L[U_n((s,v),y)] = U_n((s,v),y)$, thus

$$U_n((s,v),y) - \chi_n U_{n-1}((s,v),y) = hD_{n-1}\left(U(s,v) - \frac{v^n}{s^n}\sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0) + \frac{v^n}{s^n}S\{N(u(x,y))\} + \frac{v^n}{s^n}S\{Qu(x,y)\} - \frac{v^n}{s^n}S\{f(x,y)\}\right) \quad (7)$$

With the application of Homotopy derivative D_{n-1}, (7) becomes

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$$U_n\left((s,v),y\right) - \chi_n U_{n-1}\left((s,v),y\right) = h\left(U(s,v) - (1-\bar{\chi}_{n-1})\left(\frac{v^n}{s^n}\sum_{j=0}^{n-1}\left(\frac{s}{v}\right)^{n-(j+1)} u^{(j)}(0) - \frac{v^n}{s^n}S\{f(x,y)\}\right) + D_{n-1}\left[\frac{v^n}{s^n}S\{N(u(x,y))\} + \frac{v^n}{s^n}S\{Qu(x,y)\}\right]\right), \quad (8)$$

where

$$\chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases} \quad \text{and}$$

$$\bar{\chi}_{n-1} = \begin{cases} 0, & n-1 < 1 \\ 1, & n-1 \ge 1 \end{cases}$$

6. Application of Shehu Transform Homotopy Analysis Method (STHAM) to Selected Problems in Nonlinear PDEs and ODEs

Problem 1 [12]

Use STHAM to solve the nonlinear PDE

$u_{xx} + u^2 - u_{yy}^2 = 0, \quad u(0,y) = 0, \quad u_x(0,y) = \cos(y),$

where $u = u(x,y)$.

Solution

$$\frac{\partial^2 u}{\partial x^2} + u^2 - \left(\frac{\partial^2 u}{\partial y^2}\right)^2 = 0$$

We shall take Shehu transform of (i) to get

$$S\left\{\frac{\partial^2 u}{\partial x^2}\right\} + S\{u^2\} - S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\} = S\{0\}$$

But

$$S\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{s^2}{v^2}U\left((s,v),y\right) - \frac{s}{v}u(0,y) - u_x(0,y)$$

Substituting (iii) in (ii), we have

$$\frac{s^2}{v^2}U\left((s,v),y\right) - \frac{s}{v}u(0,y) - u_x(0,y) + S\{u^2\} - S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\} = 0$$

$$\frac{s^2}{v^2}U\left((s,v),y\right) - \frac{s}{v}.0 - \cos(y) + S\{u^2\} - S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\} = 0$$

Dividing through by $\frac{s^2}{v^2}$ yields

$$U\left((s,v),y\right) - \frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\{u^2\} - \frac{v^2}{s^2}S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\} = 0$$

Hence, the general nonlinear term is

$$N[U((s,v),y;q)] = U\left((s,v),y\right) - \frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\{u^2\} - \frac{v^2}{s^2}S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\},$$

and the linear operator

$$L[U((s,v),y;q)] = U((s,v),y).$$

Using the nth order deformation equation, we have

$$U_n\left((s,v),y\right) - \chi_n U_{n-1}\left((s,v),y\right) = hD_{n-1}[U\left((s,v),y\right) - \frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\{u^2\} - \frac{v^2}{s^2}S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\}]$$

$$U_n\left((s,v),y\right) = \chi_n U_{n-1}\left((s,v),y\right) + hD_{n-1}[U\left((s,v),y\right) - \frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\{u^2\} - \frac{v^2}{s^2}S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\}]$$

$$U_n\left((s,v),y\right) = U_{n-1}\left((s,v),y\right) + h\left(U_{n-1}\left((s,v),y\right) - (1-\bar{\chi}_{n-1})\frac{v^2}{s^2}\cos(y) + D_{n-1}\left[\frac{v^2}{s^2}S\{u^2\}\right] - D_{n-1}\left[\frac{v^2}{s^2}S\{\left(\frac{\partial^2 u}{\partial y^2}\right)^2\}\right]\right)$$

Taking $h = -1$ and applying homotopy derivative rules, we get

$$U_n\left((s,v),y\right) = -(1-\chi_n)U_{n-1}\left((s,v),y\right) + (1-\bar{\chi}_{n-1})\frac{v^2}{s^2}\cos(y) - \frac{v^2}{s^2}S\left\{\sum_{j=0}^{n-1} u_j u_{n-1-j}\right\} + \frac{v^2}{s^2}S\left\{\sum_{j=0}^{n-1}\frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y}\right\}$$

$$U_n\left((s,v),y\right) + (1-\chi_n)U_{n-1}\left((s,v),y\right) - (1-\bar{\chi}_{n-1})\frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\left\{\sum_{j=0}^{n-1} u_j u_{n-1-j}\right\} - \frac{v^2}{s^2}S\left\{\sum_{j=0}^{n-1}\frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y}\right\} = 0.$$

With χ_n and $\bar{\chi}_{n-1}$ having their usual meaning, we have for $n = 1$:

$$U_1\left((s,v),y\right) + (1-\chi_1)U_0\left((s,v),y\right) - (1-\bar{\chi}_0)\frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\{u_0 u_0\} - \frac{v^2}{s^2}S\left\{\frac{\partial^2 u_0}{\partial y^2}.\frac{\partial^2 u_0}{\partial y^2}\right\} = 0.$$

(i)

The initial approximation is obtained as:

$u_0(x,y) = u(a,b) + u_x(a,b)(x-a) + u_y(a,b)(y-b).$

(ii)

From the boundary conditions, $a = 0$ and $b = y$.

Thus

$u_0(x,y) = u(0,y) + u_x(0,y)(x-0) + u_y(0,y)(y-y).$

$u_0(x,y) = 0 + x\cos(y) + 0.u_y(0,y) = x\cos(y).$

$$U_1\left((s,v),y\right) = U_0\left((s,v),y\right) - \frac{v^2}{s^2}\cos(y) + S\{x^2\cos^2(y)\} - \frac{v^2}{s^2}S\{(-x\cos(y))^2\} = 0$$

$$U_1\left((s,v),y\right) = \frac{v^2}{s^2}\cos(y) - \frac{v^2}{s^2}\cos(y) + \frac{v^2}{s^2}S\{x^2\cos^2(y)\} - \frac{v^2}{s^2}S\{(-x\cos(y))^2\} = 0$$

$U_1\left((s,v),y\right) = 0.$

Taking inverse Shehu transform of both sides, we have

$S^{-1}\{U_1((s,v),y)\} = S^{-1}\{0\}.$

$u_1(x,y) = 0.$

The subsequent terms also give zero. Thus, the solution of the given PDE is

$u(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + \cdots.$

$u(x,y) = x\cos(y) + 0 + 0 + \cdots.$

$u(x,y) = x\cos(y).$

Problem 2 [12]

Use STHAM to solve the nonlinear PDE

$u_t + u_x^2 = 0, \quad u(x,0) = x, \quad t > 0.$

Solution

Application of Shehu transform to the given PDE yields

$$S\left\{\frac{\partial u}{\partial t}\right\}+S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}=S\{0\}.$$

But

$$S\left\{\frac{\partial u}{\partial t}\right\}=\frac{s}{v}U(x,(s,v))-u(x,0).$$

Substituting $S\left\{\frac{\partial u}{\partial t}\right\}$ back in the earlier equation yields

$$\frac{s}{v}U(x,(s,v))-u(x,0)+S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}=0.$$

$$\frac{s}{v}U(x,(s,v))-x+S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}=0$$

$$U(x,(s,v))-\frac{v}{s}x+\frac{v}{s}S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}=0.$$

Hence, the general nonlinear term is

$$N[U(x,(s,v);q)]=U(x,(s,v))-\frac{v}{s}x+\frac{v}{s}S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}$$

and the linear operator

$$L[U(x,(s,v);q)]=U(x,(s,v))$$

Using the nth order deformation equation, we have

$$L[U_n(x,(s,v))-\chi_n U_{n-1}(x,(s,v))]=hD_{n-1}[N[U(x,(s,v);q)]]$$

$$U_n(x,(s,v))-\chi_n U_{n-1}(x,(s,v))=hD_{n-1}\left[U(x,(s,v))-\frac{v}{s}x+\frac{v}{s}S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}\right]$$

$$U_n(x,(s,v))=\chi_n U_{n-1}(x,(s,v))+h\left(U_{n-1}(x,(s,v))-(1-\bar{\chi}_{n-1})\frac{v}{s}x+\frac{v}{s}D_{n-1}\left[S\left\{\left(\frac{\partial y}{\partial x}\right)^2\right\}\right]\right)$$

$$U_n(x,(s,v))=\chi_n U_{n-1}(x,(s,v))+h\left(U_{n-1}(x,(s,v))-(1-\bar{\chi}_{n-1})\frac{v}{s}x+\frac{v}{s}S\left\{\sum_{j=0}^{n-1}\frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y}\right\}\right)$$

Taking $h=-1$:

$$U_n(x,(s,v))=-(1-\chi_n)U_{n-1}(x,(s,v))+(1-\bar{\chi}_{n-1})\frac{v}{s}x-\frac{v}{s}S\left\{\frac{\partial u_0}{\partial y}\frac{\partial u_0}{\partial y}\right\}$$

$$U_n(x,(s,v))+(1-\chi_n)U_{n-1}(x,(s,v))-(1-\bar{\chi}_{n-1})\frac{v}{s}x+\frac{v}{s}S\left\{\frac{\partial u_0}{\partial y}\frac{\partial u_0}{\partial y}\right\}=0.$$

The notations χ_n and $\bar{\chi}_{n-1}$ have their usual meanings, thus for $n=1$:

$$U_1(x,(s,v))+(1-\chi_1)U_0(x,(s,v))-(1-\bar{\chi}_1)\frac{v}{s}x+\frac{v}{s}S\left\{\frac{\partial u_0}{\partial y}\frac{\partial u_0}{\partial y}\right\}=0$$

$$U_1(x,(s,v))+\frac{v}{s}x-\frac{v}{s}x+\frac{v}{s}S\{1.1\}=0$$

$$U_1(x,(s,v))+\frac{v}{s}\left(\frac{v}{s}\right)=0$$

$$U_1(x,(s,v))=-\frac{v^2}{s^2}$$

Taking inverse Shehu transform of both sides

$$u_1(x,t)=-t.$$

For $n=2$:

$$U_2(x,(s,v))+(1-\chi_2)U_1(x,(s,v))-(1-\bar{\chi}_1)\frac{v}{s}x+\frac{v}{s}S\left\{\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y}+\frac{\partial u_1}{\partial y}\frac{\partial u_0}{\partial y}\right\}=0$$

$$U_2(x,(s,v))+\frac{v}{s}S\{1.0+0.1\}=0$$

$$U_2(x,(s,v))+\frac{v}{s}.0=0$$

$$U_2(x,(s,v))=0$$

Taking the inverse Shehu transform of both sides we have

$$S^{-1}\{U_2(x,(s,v))\}=S^{-1}\{0\}.$$

$$u_2(x,t)=0.$$

For $n\geq 3$, the equation reduces to:

$$U_n(x,(s,v))+\frac{v}{s}S\left\{\sum_{j=0}^{n-1}\frac{\partial u_j}{\partial y}\frac{\partial u_{n-1-j}}{\partial y}\right\}=0.$$

These yields

$$u_n(x,t)=0,\ \text{for}\ n\geq 3.$$

Thus, the solution to the PDE is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots.$$

$$u(x,t)=x-t+0+0+0+\cdots.$$

$$u(x,t)=x-t.$$

Problem 3 [14]

Use the STHAM to solve the nonlinear IVP

$$y''(x)-(y'(x))^2+(y(x))^2=1,\quad y(0)=1,\quad y'(0)=0$$

Taking Shehu transform of the IVP in (1), we have

$$S\{y''(x)\}-S\{(y'(x))^2\}+S\{(y(x))^2\}=S\{1\}$$

$$\frac{s^2}{u^2}Y(s,u)-\frac{s}{u}y(0)-y'(0)-S\{(y'(x))^2\}+S\{(y(x))^2\}=\frac{u}{s}$$

$$Y(s,u)-\frac{u}{s}-\frac{u^2}{s^2}S\{(y'(x))^2\}+\frac{u^2}{s^2}S\{(y(x))^2\}=\frac{u^3}{s^3}$$

$$Y(s,u)-\left(\frac{u}{s}+\frac{u^3}{s^3}\right)-\frac{u^2}{s^2}S\{(y'(x))^2\}+\frac{u^2}{s^2}S\{(y(x))^2\}=0$$

The nonlinear operator is given as

$$N[\bar{\phi}((s,u);q)]=Y(s,u)-\left(\frac{u}{s}+\frac{u^3}{s^3}\right)-\frac{u^2}{s^2}S\{(y'(x))^2\}+\frac{u^2}{s^2}S\{(y(x))^2\}$$

and the linear operator

$$L[\bar{\phi}((s,u);q)]=Y(s,u),$$

where $q\epsilon[0,1]$ and $\emptyset((s,u);q)$ is a function of (s,u) and q.

Now, we make use of the nth order deformation equation

$$L[Y_n(s,u)-\chi_n Y_{n-1}(s,u)]=hD_{n-1}[N[\bar{\phi}((s,u);q)]]$$

$$Y_n(s,u)=\chi_n Y_{n-1}(s,u)+h[Y_{n-1}(s,u)-(1-\chi_{n-1})(\frac{u}{s}+\frac{u^3}{s^3})-\frac{u^2}{s^2}S\{D_{n-1}[(y'(x))^2]\}+\frac{u^2}{s^2}S\{D_{n-1}[(y(x))^2]\}]$$

Taking $h=-1$, we have, using the theorem on homotopy derivatives

$$Y_n(s,u)=\chi_n Y_{n-1}(s,u)-Y_{n-1}(s,u)+(1-\chi_{n-1})\left(\frac{u}{s}+\frac{u^3}{s^3}\right)+\frac{u^2}{s^2}S\{\sum_{j=0}^{n-1}y_j'y_{n-1-j}'\}-\frac{u^2}{s^2}S\left\{\sum_{j=0}^{n-1}y_jy_{n-1-j}\right\}$$

$$Y_n(s,u)=-(1-\chi_n)Y_{n-1}(s,u)+(1-\chi_{n-1})\left(\frac{u}{s}+\frac{u^3}{s^3}\right)+\frac{u^2}{s^2}S\{\sum_{j=0}^{n-1}y_j'y_{n-1-j}'\}-\frac{u^2}{s^2}S\left\{\sum_{j=0}^{n-1}y_jy_{n-1-j}\right\}$$

$$Y_n(s,u)+(1-\chi_n)Y_{n-1}(s,u)-(1-\chi_{n-1})\left(\frac{u}{s}+\frac{u^3}{s^3}\right)-\frac{u^2}{s^2}S\left\{\sum_{j=0}^{n-1}y_j'y_{n-1-j}'\right\}+\frac{u^2}{s^2}S\left\{\sum_{j=0}^{n-1}y_jy_{n-1-j}\right\}=0,$$

with χ_n and χ_{n-1} having their usual meaning.

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For $n = 1$:

$$Y_1(s,u) + (1-\chi_1)Y_0(s,u) - (1-\chi_0)\left(\frac{u}{s}+\frac{u^3}{s^3}\right) - \frac{u^2}{s^2}S\{y_0'y_0'\} + \frac{u^2}{s^2}S\{y_0y_0\} = 0.$$

Also, from the initial conductions,

$y_0(x) = y(0) + xy'(0) = 1 + x.0 = 1.$

$$Y_1(s,u) + Y_0(s,u) - \frac{u}{s} - \frac{u^3}{s^3} - \frac{u^2}{s^2}S\{0.0\} + \frac{u^2}{s^2}S\{1\} = 0$$

$$Y_1(s,u) + \frac{u}{s} - \frac{u}{s} - \frac{u^3}{s^3} + \frac{u^2}{s^2}.\frac{u}{s} = 0.$$

Thus,

$Y_1(s,u) = 0.$

Taking inverse Shehu transform of both sides,

$S^{-1}\{Y_1(s,u)\} = S^{-1}\{0\}.$

$y_1(x) = 0.$

For $n = 2$:

$$Y_2(s,u) + (1-\chi_2)Y_1(s,u) - (1-\chi_1)\left(\frac{u}{s}+\frac{u^3}{s^3}\right) - \frac{u^2}{s^2}S\{2y_0'y_1'\} + \frac{u^2}{s^2}S\{2y_0y_1\} = 0$$

$Y_2(s,u) = 0.$

Taking inverse Shehu transform of both sides, we have

$S^{-1}\{Y_2(s,u)\} = S^{-1}\{0\}.$

Thus, $y_n(x) = 0$, for $n \geq 1$. Hence, the exact solution is

$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots.$

$y(x) = 1 + 0 + 0 + \cdots.$

Thus,

$y(x) = 1.$

7. DISCUSION OF RESULTS

The applications of both Homotopy Analysis Method and Shehu Transform Homotopy Analysis Method (STHAM) to nonlinear ordinary differential equation and partial differential give excellent results as the two give the exact solutions. The advantage of STHAM over the HAM is the reduction in computation after the first iteration of the solution. This situation encountered in the proposed method is like the ADM where whenever noise terms are encountered, the result of the first iteration gives the exact solution.

The nonlinear ODE problems considered were solved in Hermann and Saravi (2016) by using ADM, while the problems on nonlinear PDE were obtained from Wazwaz (2009) where the author also used ADM and VIM as the methods of solution. All our solutions tally with the results arrived at in the cited literatures. The proposed method does not required derivation of special polynomials, unlike ADM, and no derivation of optimal values of Langrange is required, unlike VIM.

CONCLUSION

The derivation of STHAM and its application to both initial value ODEs and PDEs have been presented. The beauty in application of the proposed method is that of drastic reduction in the volume of computation works involved. Irrespective of the strength of nonlinearities involved, the proposed method guarantees the exact solution.

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