

METHOD OF VARIATION OF PARAMETER FOR ALMOST CRITICALLY DAMPED NONLINEAR SYSTEMS

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Abstract: Second order nonlinear differential systems modeling almost non-oscillatory processes have been considered. A new perturbation technique based on the work of Krylov-Bogoliubov-Mitropolskii method has been developed to find approximate solutions for almost critically damped nonlinear systems. The solution shows a good agreement with the numerical solution.

Keywords: Almost critically damped nonlinear systems, perturbation technique, Runge-Kutta procedure.

1. Introduction

Krylov and Bogoliubov [1] used a perturbation method to discuss transients in equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad (1.1)$$

where over-dots denote differentiation with respect to t , ω_0 is a positive constant and ε is a small parameter. This method was amplified and justified by Bogoliubov and Mitropolskii [2] and later extended by Popov [3] to the following damped oscillatory system

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}), \quad (1.2)$$

where $k > 0$ and $\omega > 0$. Mendelson [4] rediscovered the Popov's results. Today the method is well known as Krylov-Bogoliubov-Mitropolskii (KBM) method in the theory of nonlinear oscillations. Murty, Deekshatulu and Krisna [5] used the KBM method to discuss transients in equation (1.2) for the over-damped case, i.e., for $k > \omega$. Murty [6] presented a unified KBM method for solving equation (1.2). Sattar [7] found a solution of (1.2) characterized by critical damping, i.e., for $k = \omega$. Alam [8] extended the unified method of

Murty [6] to critically damped nonlinear systems. Alam [9] also found asymptotic solution of (1.2) when the unperturbed equation (or linear equation) of (1.2) has two complex roots, $-k \pm i\omega_0$ where $\omega_0^2 = \omega^2 - k^2$ and $\omega_0 \leq k < \omega$. In this case Popov's or Mendelson's solution does not give desired results. The solution obtained by Popov or Mendelson gives desired results when $k < \omega_0$. The solutions obtained by Alam [8,9] does not also give desired results when $\omega_0^2 = O(\varepsilon)$.

The aim of the present paper is to obtain a solution of (1.2) following the same perturbation method used in [1,2] when the unperturbed equation (1.2) has two complex roots, $-k \pm i\omega_0$ and $\omega_0^2 = O(\varepsilon)$.

2. The method of solution

When $\varepsilon = 0$, the solution of (1.2) is

$$x(t, 0) = e^{-kt} \left(a_0 \cos \omega_0 t + b_0 \frac{\sin \omega_0 t}{\omega_0} \right), \quad (2.1)$$

where a_0 and b_0 are arbitrary constants. It is obvious that (2.1) is valid for small values of ω_0 as well as for the limit $\omega_0 \rightarrow 0$. It is noted that when $\omega_0 \rightarrow 0$, $x(t, 0) = e^{-kt} (a_0 + b_0 t)$.

Now we seek a solution of (1.2) that reduces to (2.1) as the limit $\varepsilon \rightarrow 0$. We look for a solution of (1.2) in the form :

$$x(t, \varepsilon) = e^{-kt} \left(a(t) \cos \omega_0 t + b(t) \frac{\sin \omega_0 t}{\omega_0} \right) + \varepsilon u_1(a, b, t, \omega_0) + \varepsilon^2 u_2(a, b, t, \omega_0) + \dots, \quad (2.2)$$

where a and b are functions of t , defined by the first order differential equations

$$\left. \begin{aligned} \dot{a} &= \varepsilon A_1(a, b, t, \omega_0) + \varepsilon^2 A_2(a, b, t, \omega_0) + \dots, \\ \dot{b} &= \varepsilon B_1(a, b, t, \omega_0) + \varepsilon^2 B_2(a, b, t, \omega_0) + \dots. \end{aligned} \right\} \quad (2.3)$$

Now differentiating (2.2) twice with respect to t and using relations (2.3), we obtain

$$\begin{aligned}\dot{x} = e^{-kt} & \left((-ka + b) \cos \omega_0 t - (\omega_0^2 a + kb) \times \frac{\sin \omega_0 t}{\omega_0} \right) \\ & + \varepsilon \left(e^{-kt} \left(A_1 \cos \omega_0 t + B_1 \times \frac{\sin \omega_0 t}{\omega_0} \right) + \frac{\partial u_1}{\partial t} \right) + \dots\end{aligned}\quad (2.4a)$$

$$\begin{aligned}\ddot{x} = e^{-kt} & \left((k^2 a - \omega_0^2 a - 2kb) \cos \omega_0 t + (2k\omega_0^2 a + k^2 b - \omega_0^2 b) \times \frac{\sin \omega_0 t}{\omega_0} \right) \\ & + \varepsilon \left(e^{-kt} \left(\left(\frac{\partial A_1}{\partial t} - 2hA_1 + 2B_1 \right) \cos \omega_0 t + \left(-2\omega_0^2 A_1 + \frac{\partial B_1}{\partial t} - 2hB_1 \right) \times \frac{\sin \omega_0 t}{\omega_0} \right) + \frac{\partial^2 u_1}{\partial t^2} \right) + \dots\end{aligned}\quad (2.4b)$$

Now substituting the values of \dot{x} and \ddot{x} respectively from (2.4a) and (2.4b), and x from (2.2) in (1.2) and comparing the coefficients of various powers of ε , we get for the coefficient of ε :

$$\begin{aligned}e^{-kt} & \left(\left(\frac{\partial A_1}{\partial t} + 2B_1 \right) \cos \omega_0 t + \left(-2\omega_0^2 A_1 + \frac{\partial B_1}{\partial t} \right) \times \frac{\sin \omega_0 t}{\omega_0} \right) \\ & + \frac{\partial^2 u_1}{\partial t^2} + 2k \frac{\partial u_1}{\partial t} + \omega^2 u_1 = -f^{(0)}(a, b, t, \omega_0),\end{aligned}\quad (2.5)$$

where, $f^{(0)} = f(x_0, \dot{x}_0)$ and $x_0 = e^{-kt} \left(a \cos \omega_0 t + b \frac{\sin \omega_0 t}{\omega_0} \right)$.

Usually, equation (2.5) is solved for the unknown functions A_1 , B_1 and u_1 under the assumption that u_1 does not contain first harmonic terms. This assumption is not valid when ω_0 is

small. When $\omega_0 \rightarrow 0$ or, $k \rightarrow \omega$, equation (1.2) represents the critically damped motion. In this case formula equation (2.5) takes the simplest form:

$$e^{-kt} \left(\frac{\partial A_1}{\partial t} + 2B_1 + t \frac{\partial B_1}{\partial t} \right) + \frac{\partial^2 u_1}{\partial t^2} + 2k \frac{\partial u_1}{\partial t} + k^2 u_1 = -f^{(0)}(a, b, t) \quad (2.6)$$

Alam [9] found equation (2.6) to determine the critically damped solution of (1.2) in which $k = \omega$. In [9], $f^{(0)}$ of equation (2.6) is expanded in a

$\omega_0 \rightarrow 0$, our solution reduces to that obtained in [9].

Maclaurin series, $f^{(0)} = \sum_{r=0}^{\infty} g_r t^r$ and A_1, B_1, u_1 were determined by assuming that u_1 excludes the terms of t^0 and t^1 of $f^{(0)}$.

In this article we find the solution of (1.2) under the condition that the values of k and ω are very close together, but not equal, i.e., ω_0 is much smaller rather than 1. However, if we take the limit

3. Example

As an example of the above procedure we may consider the *Duffing's* equation with a large linear damping

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3. \quad (3.1)$$

Here,

$$\begin{aligned}f^{(0)} &= e^{-3kt} \left(a^3 \cos^3 \omega_0 t + 3a^2 b \cos^2 \omega_0 t \frac{\sin \omega_0 t}{\omega_0} + 3ab^2 \cos \omega_0 t \left(\frac{\sin \omega_0 t}{\omega_0} \right)^2 + b^3 \left(\frac{\sin \omega_0 t}{\omega_0} \right)^3 \right) \\ &= e^{-3kt} \left(a^3 \cos \omega_0 t + 3a^2 b \frac{\sin \omega_0 t}{\omega_0} + \left(\frac{3ab^2}{\omega_0^2} - a^3 \right) \cos \omega_0 t \sin^2 \omega_0 t \right. \\ &\quad \left. + \left(\frac{b^3}{\omega_0^3} - \frac{3a^2 b}{\omega_0} \right) \sin^3 \omega_0 t \right).\end{aligned}\quad (3.2)$$

Substituting $f^{(0)}$ from (3.2) into (2.5), we obtain the following equations for A_1 , B_1 and u_1

$$\frac{\partial A_1}{\partial t} + 2B_1 = -a^3 e^{-2kt}, \quad (3.3)$$

$$-2\omega_0^2 A_1 + \frac{\partial B_1}{\partial t} = -3a^2 b e^{-2kt}, \quad (3.4)$$

and

$$\frac{\partial^2 u_1}{\partial t^2} + 2k \frac{\partial u_1}{\partial t} + \omega^2 u_1 = -e^{-3kt} \left(\left(\frac{3ab^2}{\omega_0^2} - a^3 \right) \cos \omega_0 t \sin^2 \omega_0 t + \left(\frac{b^3}{\omega_0^3} - \frac{3a^2 b}{\omega_0} \right) \sin^3 \omega_0 t \right). \quad (3.5)$$

Equations (3.3) and (3.4) are two simultaneous differential equations. Their particular solutions give the unknown functions A_1 and B_1 . Substituting

$$A_1 = e^{-2kt} (l_1 a^3 + l_2 a^2 b),$$

$$\text{and } B_1 = e^{-2kt} (m_1 a^3 + m_2 a^2 b)$$

into (3.3) and (3.4) and equating the coefficients of $a^3 e^{-2kt}$ and $a^2 b e^{-2kt}$, we obtain four algebraic equations as :

$$\begin{aligned} -2kl_1 + 2m_1 &= -1, & -2\omega_0^2 l_1 - 2km_1 &= 0, \\ -2kl_2 + 2m_2 &= 0, & -2\omega_0^2 l_2 - 2km_2 &= -3. \end{aligned} \quad (3.6)$$

The solution of (3.6) is

in accordance with Alam's [8] assumptions:

$$\begin{aligned} l_1 &= \frac{k}{2(k^2 + \omega_0^2)} = \frac{k}{2\omega^2}, \\ m_1 &= \frac{-\omega_0^2}{2(k^2 + \omega_0^2)} = \frac{-\omega_0^2}{2\omega^2}, \\ l_2 &= \frac{3}{2(k^2 + \omega_0^2)} = \frac{3}{2\omega^2}, \\ m_2 &= \frac{3k}{2(k^2 + \omega_0^2)} = \frac{3k}{2\omega^2}. \end{aligned} \quad (3.7)$$

Thus the particular solutions of (3.3) and (3.4) are

$$\begin{aligned} A_1 &= \frac{a^2 (ka + 3b) e^{-2kt}}{2\omega^2}, \\ B_1 &= \frac{a^2 (-\omega_0^2 a + 3kb) e^{-2kt}}{2\omega^2}. \end{aligned} \quad (3.8)$$

By assuming a, b as constants (see [10]) in the right hand side of Eq.(3.5), we can easily determine the particular solution of this equation as :

$$\begin{aligned} u_1 &= \frac{e^{-3kt} \left(a^3 - \frac{3ab^2}{\omega_0^2} \right)}{16(k^2 + \omega_0^2)} \left(\frac{k \cos \omega_0 t - \omega_0 \sin \omega_0 t}{k} - \frac{(k^2 - 2\omega_0^2) \cos 3\omega_0 t - 3k\omega_0 \sin 3\omega_0 t}{k^2 + 4\omega_0^2} \right) \\ &+ \frac{e^{-3kt} \left(\frac{3a^2 b}{\omega_0} - \frac{b^3}{\omega_0^3} \right)}{16(k^2 + \omega_0^2)} \left(\frac{3\omega_0 \cos \omega_0 t + 3k\omega_0 \sin \omega_0 t}{k} - \frac{3k\omega_0 \cos 3\omega_0 t + (k^2 - 2\omega_0^2) \sin 3\omega_0 t}{k^2 + 4\omega_0^2} \right). \end{aligned} \quad (3.9)$$

Substituting the values of A_1 and B_1 from (3.8) into (2.3), we integrate them with respect to t , and under the assumption that a and b are constants (since the change of these variables is small as ε is, see [10] for details) in the right hand sides of (2.3), we obtain the following results (as first approximation) :

$$\begin{aligned} a &= a_0 + \frac{\varepsilon a_0^2 (ka_0 + 3b_0)(1 - e^{-2kt})}{4k\omega^2}, \\ b &= b_0 + \frac{\varepsilon a_0^2 (-\omega_0^2 a_0 + 3kb_0)(1 - e^{-2kt})}{4k\omega^2}. \end{aligned} \quad (3.10)$$

Therefore, the first approximate solution of (3.1) is obtained as:

$$x = e^{-kt} (a \cos \omega_0 t + b \sin \omega_0 t / \omega_0) + \varepsilon u_1, \quad (3.11)$$

where a, b and u_1 are given by respectively (3.10) and (3.9). The method can be carried on to higher orders in a similar way.

4. Initial conditions

The solution can be used in a general initial value problem. Usually, $[x(0), \dot{x}(0)]$ is specified.

We can calculate the initial values of a and b , i.e., a_0 and b_0 by solving the following equations :

$$x(0) = a + \frac{\varepsilon}{16\omega^2} \left[\left(1 - \frac{k^2 - 2\omega_0^2}{k^2 + 4\omega_0^2} \right) \left(a^3 - \frac{3ab^2}{\omega_0^2} \right) + \left(\frac{3\omega_0}{k} - \frac{3k\omega_0}{k^2 + 4\omega_0^2} \right) \left(\frac{3a^2b}{\omega_0} - \frac{b}{\omega_0^3} \right) \right], \quad (4.1a)$$

$$\dot{x}(0) = -ka + b + \frac{\varepsilon}{16\omega^2} \left[-3k + \left(\frac{-\omega_0^2}{k} + \frac{9k\omega_0^2}{k^2 + 4\omega_0^2} \right) \left(a^3 - \frac{3ab^2}{\omega_0^2} \right) + \left(3\omega_0^2 - \frac{9k\omega_0^2}{k^2 + 4\omega_0^2} \right) \right]. \quad (4.1b)$$

Equations (4.1a) and (4.1b) are nonlinear simultaneous equations (algebraic). In general, a numerical method (mainly Newton-Raphson) is used to solve these equations (see [10-11]).

5. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we compare the approximate solution with the numerical solution (considered exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer to the work of Murty, Deekshatulu and Krisna [5]. In our paper, for different damping forces we have compared the analytic solution (3.11) for $\varepsilon = 0.1$ with those obtained by *Runge-Kutta* fourth-order procedure.

First of all, $x(t)$ has been computed using the solution (3.11) with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, for $k = \sqrt{0.9}$, $\omega = 1$, i.e., for $\omega_0^2 = 0.1$. Then the numerical solution (by *Runge-Kutta* procedure) has been obtained and the percentage errors have been calculated. All the results are shown in Table 1. From Table 1, it is seen that for most of the times the errors of the results obtained from (3.11) are less than 1%. Thus the solution and the numerical one are almost identical up to an accuracy of ε^2 . To

compare the new solution with Alam's [9] existing solution we have calculated it for the same initial conditions (as well as same values of k and ω) and presented in the fourth column (with percentage errors in fifth column) of Table 1. Comparing the errors of both solutions we conclude that both solutions are useful for this case.

Next we calculate both solutions for the case $k = \sqrt{0.95}$, $\omega_0^2 = 0.05$. Here we consider the same initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. All the results are shown in Table 2. From Table 2, it is clear that the errors of our new solution (for most of the times) are less than Alam's [9] solution (note: after $t = 1.5$, the error of solution of [9] is more than 1%, but theoretically it should be 1% or less than 1% as it is chosen that $\varepsilon = 0.1$). Thus the new solution is useful near the critical damping. Similarly, we can compare the new solution with the critically damped solution obtained in [8]. In the later case we shall see that the solution of [8] gives better result than our solution only for $\omega_0 \rightarrow 0$.

Table 1

t	$x(t)$	x (Numerical)	Percentage Errors	x by the solution of [9]	Percentage Errors of [9]
0.0	1.000000	1.000000	0.0000	1.000000	0.0000
0.5	0.890503	0.899894	-1.0436	0.895257	-0.5153
1.0	0.699569	0.707814	-1.1649	0.702844	-0.7022
1.5	0.510640	0.515999	-1.0386	0.511841	-0.8058
2.0	0.354892	0.357933	-0.8496	0.354816	-0.8708
2.5	0.237808	0.239370	-0.6525	0.237175	-0.9170
3.0	0.154654	0.155369	-0.4602	0.153887	-0.9539
3.5	0.097934	0.098200	-0.2709	0.097232	-0.9857
4.0	0.060464	0.060512	-0.0793	0.059897	-1.0163
4.5	0.036385	0.036342	0.1183	0.035960	-1.0511
5.0	0.021304	0.021232	0.3391	0.021001	-1.0880

Table 2

t	$x(t)$	x (Numerical)	Percentage Errors	x by the solution of [9]	Percentage Errors of [9]
0.0	1.000000	1.000000	0.0000	1.000000	0.0000
0.5	0.891415	0.900624	-1.0225	0.894705	-0.6572
1.0	0.704228	0.711344	-1.0004	0.705080	-0.8805
1.5	0.519151	0.523147	-0.7638	0.517902	-1.0026
2.0	0.366211	0.368063	-0.8496	0.364095	-1.0781
2.5	0.250514	0.251175	-0.6525	0.248339	-1.1291
3.0	0.167434	0.167520	-0.0513	0.165566	-1.1664
3.5	0.109820	0.109669	0.1377	0.108356	-1.1972
4.0	0.070879	0.070660	0.3099	0.069796	-1.2228
4.5	0.045090	0.044880	0.4679	0.044321	-1.2455
5.0	0.028303	0.028129	0.6186	0.027772	-1.2692

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