BOOLEAN LATTICES WITH SECTIONAL SWITCHING MAPPINGS

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Abstract: *Consider a Boolean lattice* $L=(L,\vee,\wedge,1)$ *with a greatest element 1. An interval* $[a,1]$ *for* $a \in L$ *is called a section. In each Section* $[a,1]$ *an antitone bijection is defined. We characterize these Lattices by means of two induced binary operations providing that the resulting algebras from a variety. A mapping f, of* $[a,1]$ *on to itself is called a switching mapping if* $f(a)=1, f(1)=a$ *and for* $x \in [a,1], a \neq x \neq 1$. *We have* $a \neq f(x) \neq 1$ *If for* $p,q \in L, p \leq q$ *the mapping on the section* $[a,1]$ *is determined by that of* $[1,1]$ *, [1] it is shown that the compatibility condition is satisfied. We have got conditions for antitone of switching mapping and a connections with complementation in sections is shown.*

Keywords: *Antitone, Bijections, Section, Sectional Mapping, Compatibility Condition.*

1. Introduction

Let $L=(L,\vee,\wedge,1)$ be a Lattice with the greatest element 1. For $a \in L$, the interval $[a,1]$ will be called a section.

A mapping $f: x \alpha\ y$ is called an involution if $f(f(u))=x$ for each $x \in x$.

Let $(x,\leq)$ be an ordered set. A mapping $f: x \alpha\ y$ is antitone if, $x \leq y$ implies $f(y) \leq f(x)$ for all $x,y \in x$. A weakly switching mapping $: x \alpha\ x^1$ will be called a switching mapping if $a \neq x^a \neq 1$ for each $x \in [a,1]$ with $a \neq x \neq 1$.

We induced Lattices with 1. where for each $a \in L$ there is a mapping on the section $[a,1]$; such a structure will be called lattice with sectional mappings [1].

We study the following notation: for each $a \in L$ and $x \in [a,1]$ denote by x^a the image of x in this sectional mapping on $[a,1]$. Thus $: x \alpha\ x^a$ is a symbol for the corresponding sectional mapping on the section $[a,1]$.

Let $L=(L,\vee,\wedge,1)$ be a lattice with sectional mapping. Define the so-called induced operation on L by the rule $x \cdot y = (x \vee y)^y$. Since $x \vee y \in [y,1]$ for any $x,y \in L$. Also, conversely, if "v" is induced on L, then for each $a \in L$ and $x \in [a,1]$. We have $x \cdot a = (x \vee a)^a = x^a$.

2. Switching Mapping

A mapping $: x \alpha\ x^a$ on the section $[a,1]$ is weakly switching if $a^a = 1, 1^a = a$, in other words, a weakly switching mapping "switches" the bound element of the section.

Lemma 2.1 A lattice $L=(L,\vee,\wedge,1)$ with section involutions. The following properties are equivalent for $a \in L$

(i) $: x \alpha\ x^a$ is antitone,

(ii) The section $[a,1]$ is a Lattice where $x \wedge y = (x^a \vee y^a)^a$ (De Morgan law).

Proof. (i) $\Rightarrow (ii)$: Since the sectional mapping on $[a,1]$ is an antitone involution, it is a bijection and $x,y \leq x \vee y$ implies $x^a, y^a \geq (x \vee y)^a$ and the existence of supremum for $x,y \in [a,1]$ yields existence of the infimum $x \wedge y$.

Hence $x^a \wedge y^a \geq (x \vee y)^a$.

However, $x^a, y^a \geq x^a \wedge y^a$. Thus, due to $x = x^{aa}, y = y^{aa}$, We obtain $x,y \leq (x^a \wedge y^a)^a$. Whence

$x \vee y \leq (x^a \wedge y^a)^a$ i.e. $(x \vee y)^a \geq x^a \wedge y^a$

Al together, we obtain (ii)

(ii) ⇒ (i) : Let $x, y \in [a,1]$ and suppose $x \leq y$.

Then $x \vee y = y$ and, by (ii) $y^a = (x \vee y)^a = x^a \wedge y^a$

Thus $y^a \leq x^a$, i.e. the sectional mapping on $[a,1]$ is antitone. □

Lemma 2.2 A Lattice $L = (L, \vee, \wedge, 1)$ with sectional mappings.

(i) if the sectional mapping $: x \to x^l$ is an involution for each $I \in L$ then the induced operation satisfies the identity $(x \cdot y) \cdot y = (y \cdot x) \cdot x = x \vee y$ ....(1).

(ii) if the sectional mapping $: x \to x^l$ is weakly switching for each $1 \in L$ and the induced peration an involution satisfies (1), the every sectional mapping is an involution.

Remark 1. Identity (1) is called quasi-Commutativity in [1, 2]

Proof: (i) Since $x \cdot y \in [y,1]$

We have $x \cdot y = (x \vee y)^y \geq y$.

Thus, if the sectional mapping is an involution we conclude,

$$(x \cdot y) \cdot y = ((x \vee y)^y \vee y)^y$$
$$= (x \vee y)^{yy} = x \vee y,$$

Whence (i) is evident.

(ii) Let each sectional mapping be weakly switching, let $1 \in L$ and $x \in [1,1]$.

Then $x \vee 1 = x$ and, by (i)

$$x^{ll} = (x \cdot l) \cdot l = (l \cdot x) \cdot x$$
$$= ((l \vee x)^x \vee x)^x = (x^x \vee x)^x = (l \vee x)^x \text{ and}$$
$$= 1^x = x$$

thus $: x \to x^l$ is an involution. □

Lemma 2.3 A Lattice $L = (L, \vee, \wedge, 1)$ with sectional mappings. Let $\leq$ be its induced order. Then $x \leq y$ if and only if $x \cdot y = 1$.

Proof: If $x \leq y$, then $x \cdot y = (x \vee y)^y = y^y = 1,$

Conversely, if $x \cdot y = 1$, then $(x \vee y)^y = 1,$

Since it is a switching mapping, $x \vee y = y$, whence , $x \leq y$ □

Lemma 2.4 A Lattice $L = (L, \vee, \wedge, 1)$ with sectional weakly switching mappings.

Then L satisfies identities,

$x \cdot x = 1, 1 \cdot x = x, x \cdot 1 = 1$.......(2)

Proof: Since $x \cdot y = (x \vee y)^y$

Thus $x \cdot x = ((x \vee x)^y = y^y = 1,$

Again, since in a sectional switching mappings.

$$x^{ll} = (x \cdot l) \cdot l$$
$$= (l \cdot x) \cdot x$$
$$= ((l \vee x)^x \vee x)^x$$
$$= (x^x \vee x)^x$$
$$= (1 \vee x)^x$$
$$= 1^x = x$$
$$further, x^{ll} = (x \cdot l) \cdot l$$
$$= x \cdot x = (x \vee x)^x$$
$$= x^x$$
$$= 1. \ □$$

Theorem 2.5 A Lattice $L = (L, \vee, \wedge, 1)$ with sectional switching mappings.

(i) If L satisfies the identity $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1$.......(3)

Then every switching mapping on L is antitone.

(ii) If every sectional switching mappings. on L is an involution then it antitone if and only if L satisfies (3)

Proof: (i) Let $z \in L$, $x, y \in [z,1]$ and $x \leq y$.

By Lemma 2.3 we have $y \cdot z = 1,$ and by Lemma 2.4 and (3) we conclude:

$$(y \cdot z) \cdot (x \cdot z) = ((1 \cdot y) \cdot z) \cdot (x \cdot z)$$
$$= (((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z)$$
$$= 1.$$

By Lemma 2.3 we have $y \cdot z \leq x \cdot z$

and thus $y^z = y \cdot z \leq x \cdot z = x^z$

(ii) Let the sectional switching mappings on L are antitone involutions [2],[3],[4].

By Lemma 2.2 we have $(x \cdot y) \cdot y = x \vee y$.

Since $x \vee y \vee z \geq x \vee z$ and

$x \vee y \vee z, x \vee z \in [z,1]$

We obtain,

$$((x \vee y) \vee y) \vee z = (x \vee y \vee z)^z \leq (x \vee z)^z = x \vee z$$

By Lemma 2.3 we conclude $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1$ □

3. The Compatibility Condition

Consider a Lattice with sectional mappings where the mapping in a smaller section is determined by that of a greater one.

We say that $L = (L, \vee, \wedge, 1)$ satisfies the compatibility condition if $p \leq q \leq x$ implies that

$$x^q = x^p \vee q \text{ ....(4)}$$

It is easy to verify that (4) can be equivalently expressed as the following identity,

$(y \vee x) \vee (x \vee y) = ((y \vee x) \vee x) \vee (x \vee y)$.......(5)

Since $x \leq x \vee y \leq x \vee y \vee x$ and

$(y \vee x) \vee (x \vee y) = (x \vee y \vee x)^{(x \vee y)}$

$(y \vee x) \vee x = (x \vee y \vee x)^{x}$

Lemma 3.1 A Lattice $L = (L, \vee, \wedge)$ with sectional switching mappings, satisfying the compatibility condition. Then

(i) $x \vee x^{l} = 1$ for each $l \in L$ and each $x \in [l,1]$

(ii) If $z \mapsto z^{l}$ is a switching mappings for $l \neq 1$ then $x^{1} \neq x$ and if $x < y$ then $x^{l} \neq y^{l}$ for $x, y \in [l,1]$

(iii) If all the sectional mappings are switching, then no section of L can be a chain with more then two elements.

Proof: (i) Since, we conclude directly by (5) $1 = x^{x} = x^{l} \vee x$

(ii) If $z \mapsto z^{l}$ is a switching mapping on $[1,1]$ and $x, y \in [l,1]$, then if $x^{l} = x$, by (i),

We obtain $1 = x^{l} \vee x = x$ and, hence,

$1 = x^{l} = 1^{l} = l$, a contradiction.

If $x \leq y$ and $x^{l} = y^{l}$, then by (5) and (i),

$y^{x} = y^{l} \vee x = x^{l} \vee x = 1$

Since the sectional mapping is switching, it yields y = x, a contradiction.

(iii) Suppose that $[1,1]$ is a chain with more then two elements.

Then there exists,

$x \in [l,1], l \neq x \neq 1$

We have, $x^{l} \neq l, x^{l} \neq 1$ and by (i),

$1 = x^{l} \vee x = max(x, x^{l})$, a contradiction. □

Theorem 3.2 A Lattice $L = (L, \vee, \wedge)$ with sectional switching mapping satisfying the compatibility condition. If $x \mapsto x^{l}$ is antitone on $[1,1]$, then x^{l} is a complement of x for each $x, \in [l,1]$

Proof: Considers the sectional switching mapping on $[1,1]$ is antitone. [5]

By Lemma 2.5 we have $x \vee x^{l} = 1$ and $x^{l} \vee x^{ll} = 1$ for each $x, \in [l,1]$

Take $z = x \wedge_{l} x^{l}$

Then $z \leq x, z \leq x^{l}$ and, due to the antitone property of mapping, also $z^{l} \geq x^{l}, z^{l} \geq x^{ll}$

Thus, $z^{l} \geq x^{l} \vee x^{ll} = 1$

Therefore, it follows that,

$z^{l} = 1$,

i.e., $z = l$

and x^{l} is complement of x in the Lattice $([l,1] \vee, \wedge)$ □

Theorem 3.3 A Lattice $L = (L, \vee, \wedge)$ with sectionally antitone involutions satisfying the compatibility condition. Then for each $l \in L$ the section $[1,1]$ is an orthomodular lattice where x^{l} is an orthocomplement of $x, \in [l,1]$

Proof: Since sectionally antitone involutions are switching mappings, thus by Lemma 1.1 and Theorem 3.2, $[1,1]$ is a Lattice and x^{l} is a complement of $x, \in [l,1]$, .Since this sectional mapping is an involution, we have $x^{ll} = x$ and due to antitony, $x \leq y$ implies $y^{l} \leq x^{l}$ for $x, y \in [l,1]$, thus x^{l} is an orthocomplement of x in ,

By using the compatibility condition $l \leq x \leq y$ implies $y^{x} = y^{l} \vee x$ and hence $y \wedge_{l} (x \vee y^{l}) = y \wedge_{l} y^{x} = y \wedge_{x} y^{x} = x$ which is the orthomodular condition in the lattice $([l,1] \vee, \wedge_{l})$ □

Theorem 3.4 A Lattice $L = (L, \vee, \wedge)$ with sectionally antitone involutions. If for $l \in L$ and each $x, y \in [l,1]$, the relation

$(x^{l} \vee y) \vee x^{l} = (y^{l} \vee x)^{l} \vee y^{l}$............(6)

holds, then $([l,1] \vee, \wedge_{l})$ is a Boolean algebra.

Proof: Due to Lemma 1.1 $([l,1] \vee, \wedge_{l})$ is a lattice and we can use De Morgan law for each section. Let $a \in [l,1]$.

Using of the identity (6), we obtain

$a \vee a^{l} = a^{ll} \vee a^{l} = (a^{l} \vee l)^{l} \vee a^{l}$

$= (l^{l} \vee a)^{l} \vee l^{l} = (1 \vee a)^{l} \vee 1 = 1$

Due to the De Morgan law, we have,

$a \wedge_{l} a^{l} = a^{ll} \wedge_{l} a^{l} = (a^{l} \vee a)^{l} = 1^{l} = l.$

Hence, a^{l} is a complement of a in $[1,1]$.

Let $u \in [l,1]$, is a complement of a in $[1,1]$,

i.e. $a \vee u = 1$ and $a \wedge_{l} u = l$.

Using the identity (6) and the De Morgan law again, we derive,

$a = l \ \text{É}a = (a \ \text{É}u)^l \ \text{É}a$

$= (a^{ll} \ \text{É}u^l)^l \ \text{É}a^{ll}$

$= (u^l \ \text{É}a^l)^l \ \text{É}u^l$

$= (u \ {}_l\text{È}a) \ \text{É}u^l$

$= l \ \text{É}u^l = u^l$

Thus, $a^l = u^{ll} = u$, and the complement is unique.

Since the involution is an antitone unique complementation, then, according to [5], $([l,1] \ ,\text{É}, {}_l\text{È})$ is distributive. □

4. Conclusion

It is shown that lattice with 1 where for each $a \ L$ there is a mapping on the section $\lfloor a,1 \rfloor$. Due to this lattice with sectional mapping is considered as algebra of type $(\ ,\text{É}, \text{È})$. We also not that the compatibility condition is satisfied for complementation in any Boolean lattice and in any orthomodular lattice. Its modification holds also for lattices with sectionally antitone involutions which are implication for MV-algebra. We can take for each $a \ L$ and every $x \ [a,1]$ $a^a = 1$, $1^a = a$ and $x^a = x$ for $a \ x \ 1$.

Hence our concept is really universal and very natural for lattice theory.

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