## SECTIONALLY PSEUDOCOMPLEMENTED RESIDUAL LATTICE

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Abstract : At first, we recall the basic concept, By a residual lattice is meant an algebra  $L = (L, \lor, \land, *, \circ, 0, 1)$  such that

(i)  $L = (L, \lor, \land, 0, 1)$  is a bounded lattice,

(ii) L = (L,\*,1) is a commutative monoid,

(iii) it satisfies the so-called adjoin ness property:  $(x \lor y) * z = y$  if and only if  $y \le z \le x \circ y$ 

Let us note [7] that  $x \lor y$  is the greatest element of the set  $(x \lor y) * z = y$ 

Moreover, if we consider  $x * y = x \land y$ , then  $x \circ y$ is the relative pseudo-complement of x with respect to y, i. e., for  $* = \land$  residuated lattices are just relatively pseudo-complemented lattices. The identities characterizing sectionally pseudocomplemented lattices are presented in [3] i.e. the class of these lattices is a variety in the signature  $\{\lor,\land,\circ,1\}$ . We are going to apply a similar approach for the adjointness property:

*Key words: Residuated lattice, non Distributive, Residuated Abeliean, commutative monoid:* 

## **1. Introduction**

Residuated lattices were introduced by Ward and Dilworth [5] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [1] (where it is renamed as a residuated Abelian semi-group with a unit) and the book by R. Belohavek [7]. In this short note we will compare a certain modification of a residuated lattice with already introduced [2], [3]. At first, we recall the basic concept:

**Definition 1.** A lattice  $L = (L, \lor, \land, 1)$  with the greatest element 1 is sectionally pseudo-complemented if each interval [y, 1] is a pseudo-complemented lattice.

From now on, denote by  $x \lor y$  the pseudocomplement of  $x \lor y$  in the interval [y, 1].

Naturally,  $x \lor y \in [y,1]$  thus  $L = (L;\lor,\land,1)$  is sectionally pseudo-complemented if and only if " $\circ$ " is an (everywhere defined) operation on L.

**Definition 2.** An algebra  $L = (L; \lor, \land, *, \circ, 1)$  is called a sectionally residuated *lattice if* 

- (i)  $L = (L, \lor, \land, 0, 1)$  is a lattice with the greatest element 1;
- (ii) L = (L,\*,1) is a commutative monoid ;
- (iii) it satisfies the sectional adjointness

property:  $(x \lor y) * z = y$  if and only if  $y \le z \le x \circ y$ 

**Lemma 1.1** Let  $L = (L; \lor, \land, *, \circ, 1)$  be a sectionally residuated lattice. Then x \* y is *the greatest element of the set*  $\{z; (x \lor y) * z = y\}$ 

This immediately yields the following facts:

$$(x \lor y) * (x \circ y) = y$$
, (1)  
 $(x \lor y) * y = y$ , (2)  
 $y \le x \circ y$ , (3)

**Lemma 1.2** Let  $L = (L; \lor, \land, *, \circ, 1)$  be a sectionally residuated lattice. Then  $x \le y$ , *if and only If*  $x \circ y = 1$ 

**Proof:** Suppose  $x \le y$ , Then  $x \lor y = y$ , and by Lemma 1.1,  $x \circ y$  is the greatest element of the set  $\{z; y * z = y\}$  By Definition 2, y \* 1 = 1 thus  $x \circ y = 1$ . Conversely,

Suppose  $x \circ y = 1$  .Then, by [1], we have  $y = (x \lor y) * (x \circ y) = (x \lor y) * 1 = x \lor y$ 

whence  $x \le y$ 

**Lemma 1.3** In a sectionally residuated lattice, the following identities are satisfied:

and  $1 \circ x = x$ 

**Proof:** The first three identities follow directly by Lemma 1.2. Further, by Lemma 1.1,

 $1 \circ x$  is the greatest element of the set  $\{z; 1 * z = x\} = \{x\}$  thus  $1 \circ x = x$ 

**Lemma 1.4** In a sectionally residuated lattice, a \* b = a if and only if a = b

**Proof:** Putting x = y = a and z = b in the sectional adjointness property, the assumption a \* b = a yields  $(a \lor a) * b$  iff  $a \le b \le a \circ a = 1$  thus  $a \le b$ 

Conversely,  $a \le b$  implies by Lemma 3  $a \le b \le 1 = a \circ a$  and, by sectional adjointness,  $a * b = (a \lor a) * b = a$ 

Applying Lemma 1.2 and Lemma 1.4, we get

**Corollary 1.5** In a sectionally residuated lattice,

(a) x \* y = x if and only if  $x \circ y = 1$ ;

(b) x \* x = x

**Lemma 1.6** In a sectionally residuated lattice,  $x \land y \le x * y$ .

**Proof:** By [3] we have  $x \land y \le x \circ (x \land y)$ . Applying sectional adjointness, we infer  $x * (x \land y) = (x \lor (x \land y)) * (x \land y)$  and, analogously,  $y * (x \land y) = x \land y$ . Hence, by Corollary 1.5 (b),

$$x * y * (x \land y) = x * (x \land y) * y * (x \land y)$$

 $= (x \land y) \ast (x \land y) = x \land x$ 

and by Lemma 1.4,  $x \land y \le x * y$ .

**Theorem 1.7** Let  $L = (L; \lor, \land, *, \circ, 1)$  be a sectionally residuated lattice. Then it is a sectionally pseudo-complemented lattice.

**Proof:** Replacing y by  $x \wedge y$  in the sectional adjointness property, we obtain  $x * z = x \wedge y$  *iff*  $x \wedge y \le z \le x \circ (x \wedge y)$ .

However,  $x \circ (x \land y)$  is the greatest element of the set  $\{t; (x \lor (x \land y)) * t = x \land y\} = \{t; x * t = x \land y\}.$ 

By Lemma 1.4,  $x \wedge t \leq x * t = x \wedge y$ , thus the greatest t of this property satisfies  $t \geq y$ .

Thus  $y \le x \circ (x \land y)$ , i.e.,  $x \land y \le y \le x \circ (x \land y)$  and by the sectional adjointness,  $x * y = (x \land (x \lor y)) * y = x \land y$ . Hence,  $x \circ y$  is the pseudo-complement of

 $x \lor y$  in the interval [y,1]

## 2. Conclusion

It is well known that every relatively pseudocomplemented lattice is distributive.

An extension of relative pseudocomplementation for the non-distributive case was already involved in [3], [4]:

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